Separating quantum and classical proofs and advice with classical oracles

John Bostanci

Based on work with Jonas Haferkamp, Chinmay Nirkhe, and Mark Zhandry
and with Andrew Huang, and Vinod Vaikuntanathan

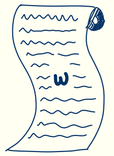
The power of proofs

Efficient Classical Algorithm

P vs NP

Unbounded prover

Classical proof



Efficient Classical Algorithm

The power of proofs

Efficient Classical Algorithm

vs

Efficient Quantum Algorithm

vs

Unbounded prover

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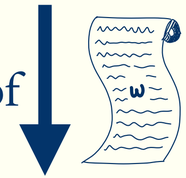

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The power of proofs

$N = pq$

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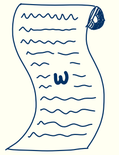

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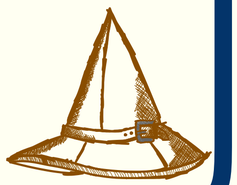

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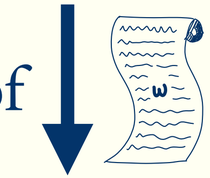

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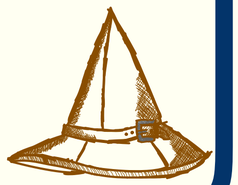
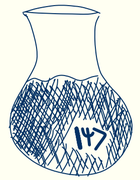


Quantum proof

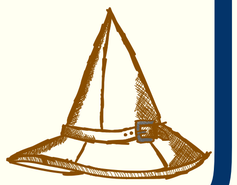


Efficient Quantum Algorithm

The power of quantum proofs

QCMA vs QMA

Unbounded prover



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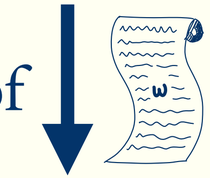

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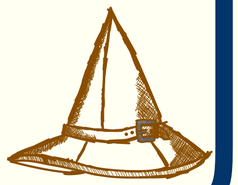

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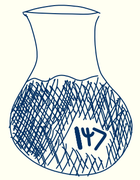

Efficient Quantum Algorithm

Quantum versus classical proofs

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→ If there were ways to check these properties with a classical proof, it would say something about the “representation complexity” of quantum systems!

→ If not, then quantum proofs really do allow us to verify something that a classical proof can’t. It will be interesting to understand why classical strings fail!

What can we say?

Unfortunately, proving an unconditional separation between the two classes would imply $\mathsf{P} \neq \mathsf{PSPACE}$, among other things.

$$\mathsf{P} \subseteq \mathsf{QCMA} \subseteq \mathsf{QMA} \subseteq \mathsf{PSPACE}$$

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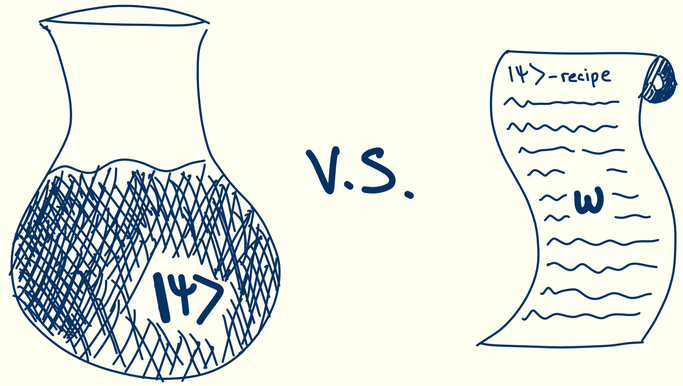
Next best thing: Oracle separation!

We will prove that there is a classical oracle relative to which QMA ≠ QCMA



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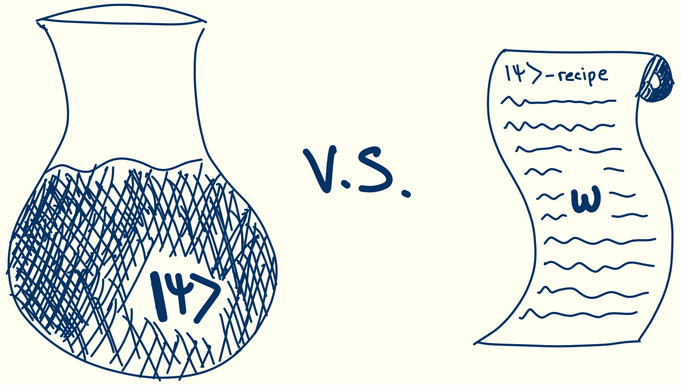

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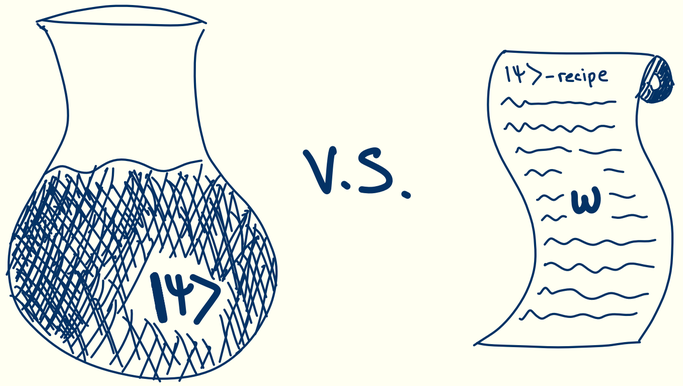

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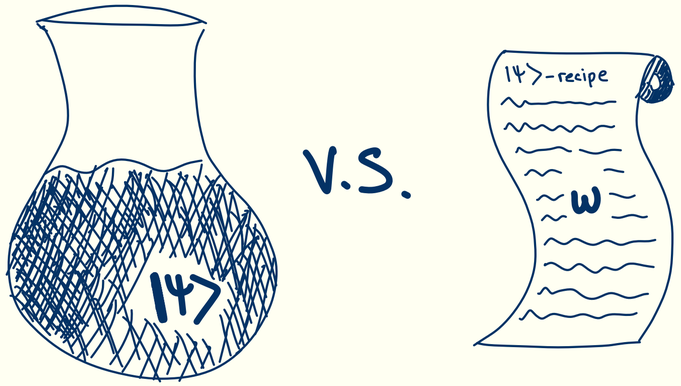
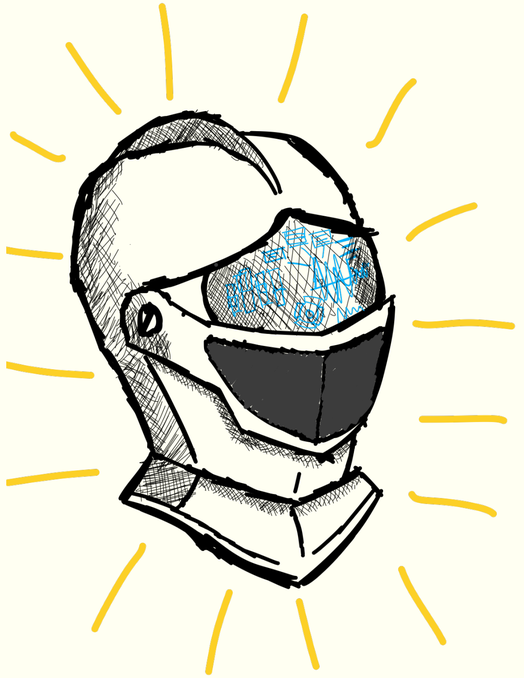


→ A quantum verifier must use their proof in an “interesting” way, can’t just measure their proof, because otherwise I could send the measurement result as a proof!



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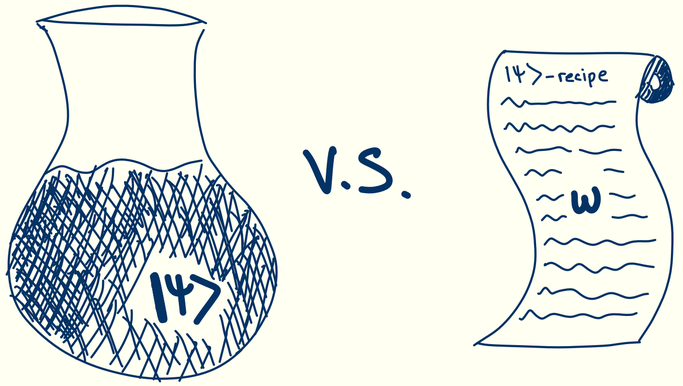

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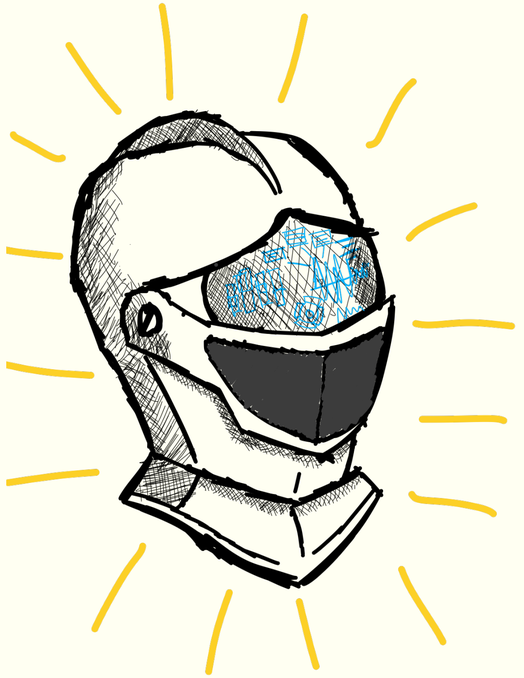
- Quantum algorithms are really good at detecting global structure, so it’s even hard to rule out BQP algorithms for some candidate oracle separations!



A new approach for ruling out QCMA algorithms

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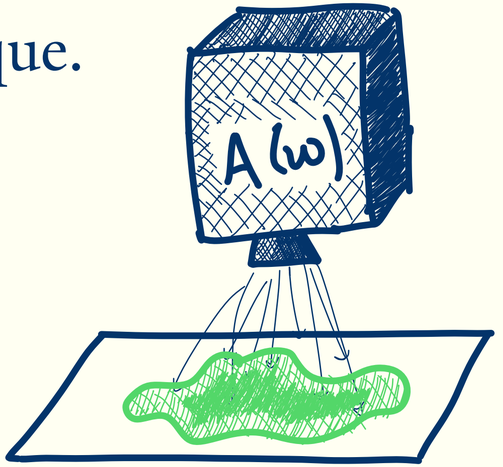
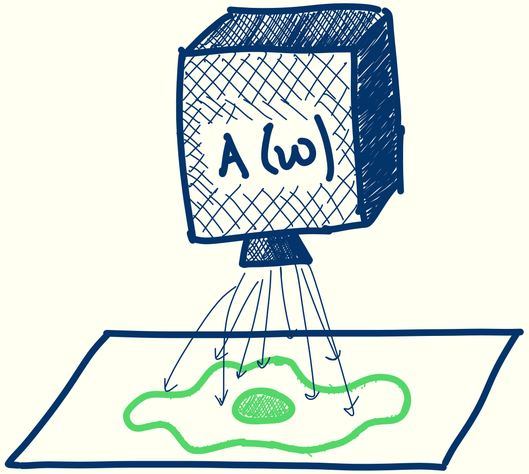
They define an oracle-input problem as follows:

Input: Oracle access to a set S

Output: Is $|S| \geq \ell$ or $|S| \leq \ell/2$, promised one of the two is the case.



VS



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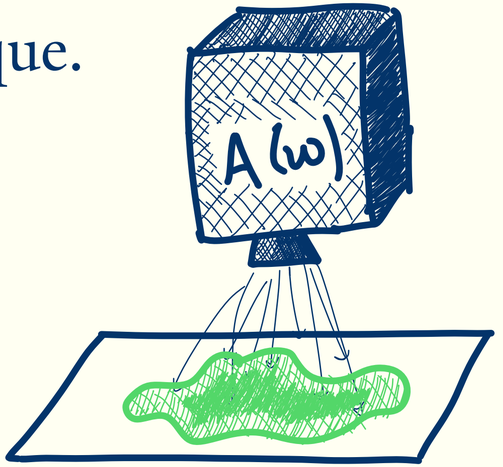

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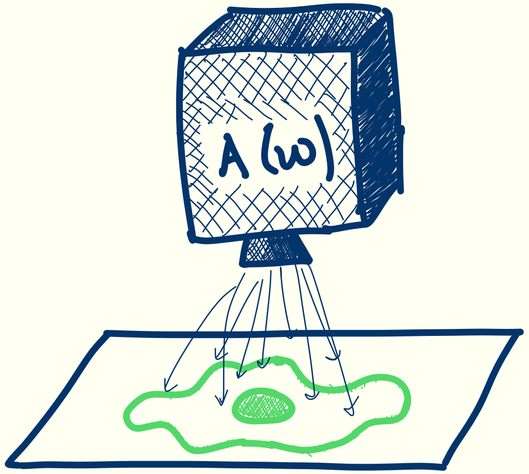

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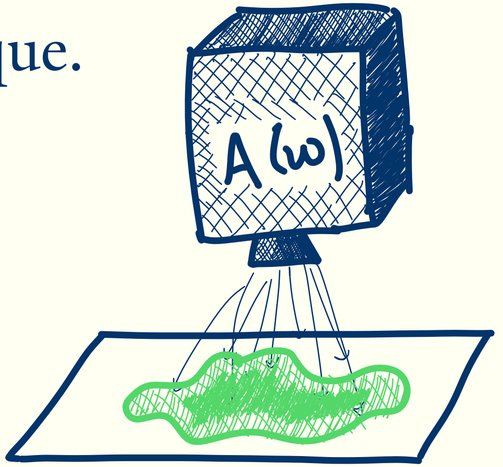

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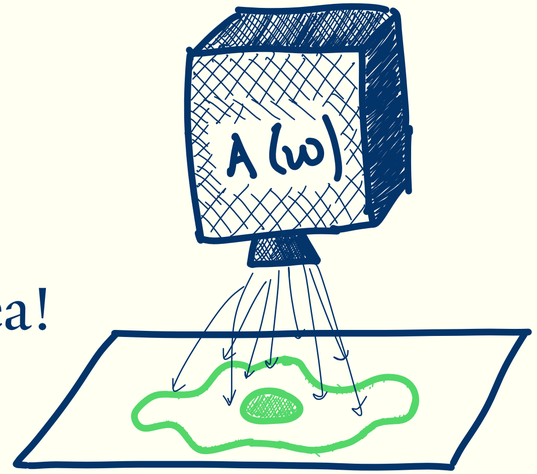

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- Fefferman and Kimmel originally approached this by fixing a "good" witness, and then applying the adversary method. Let's see another idea!

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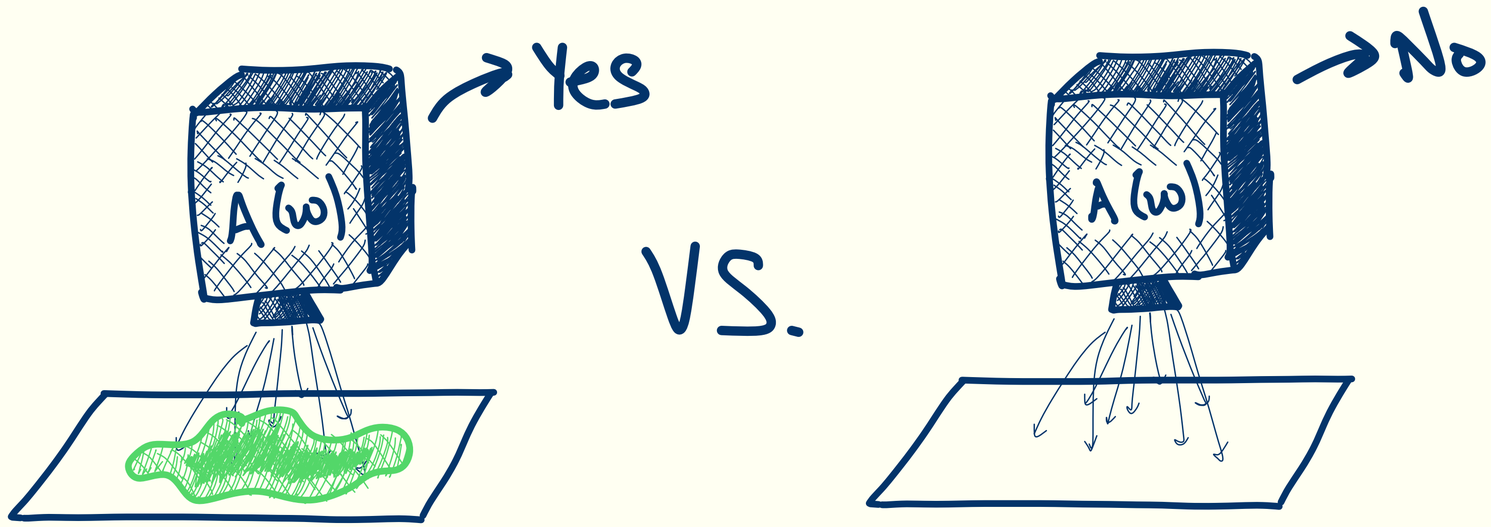

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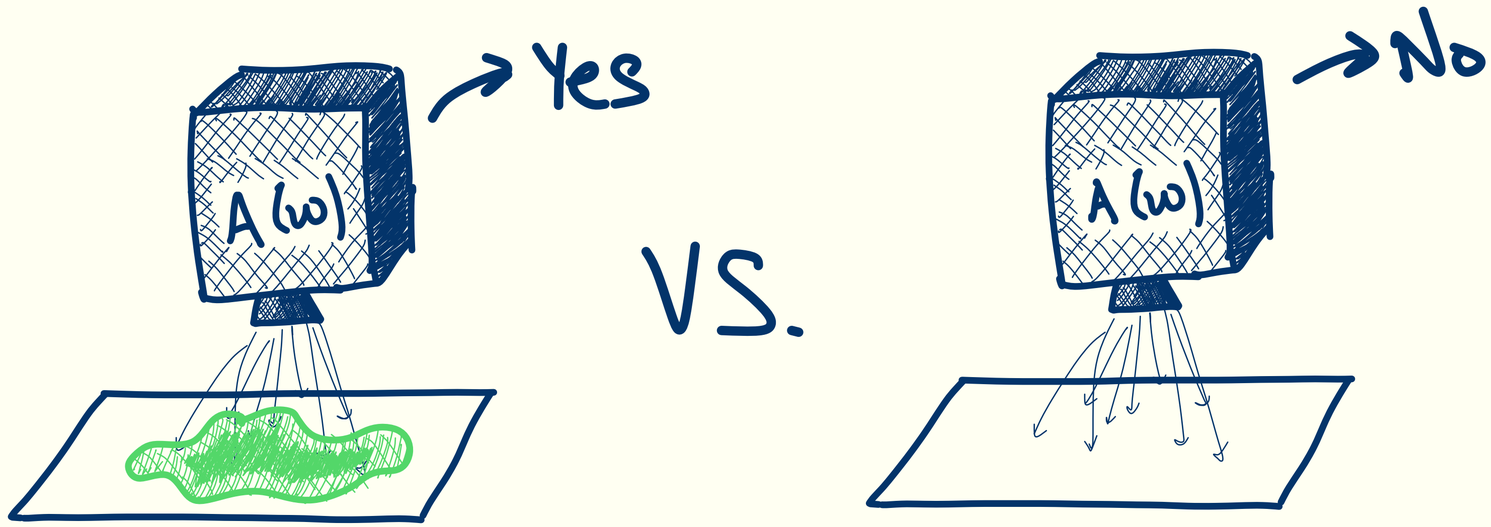

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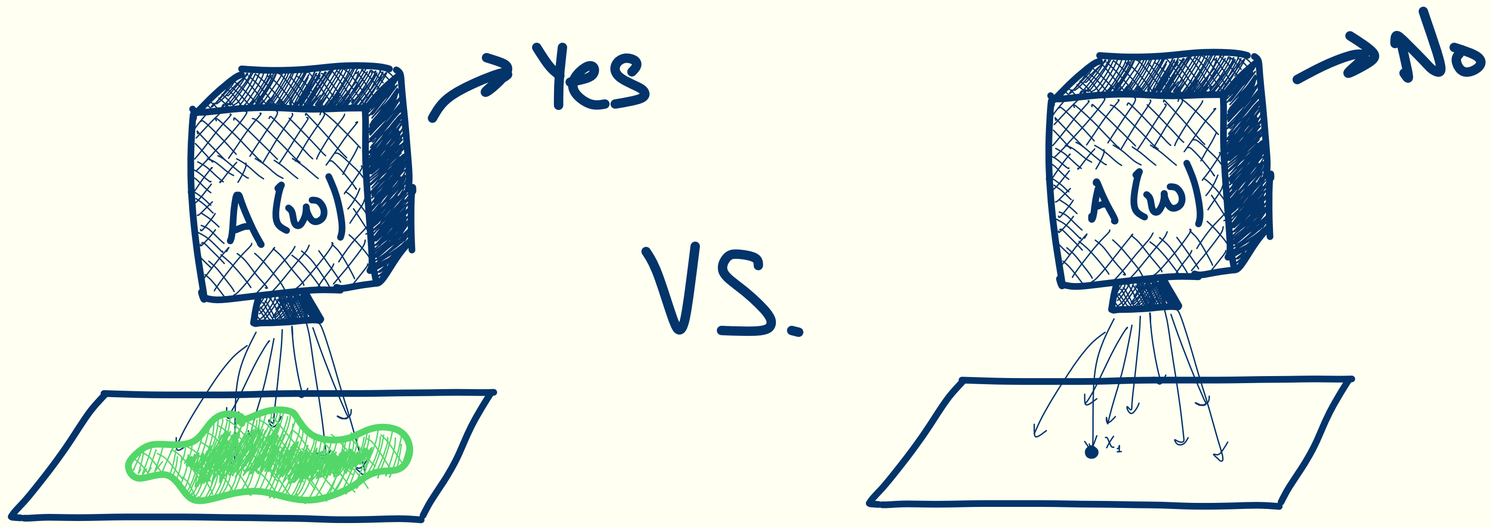

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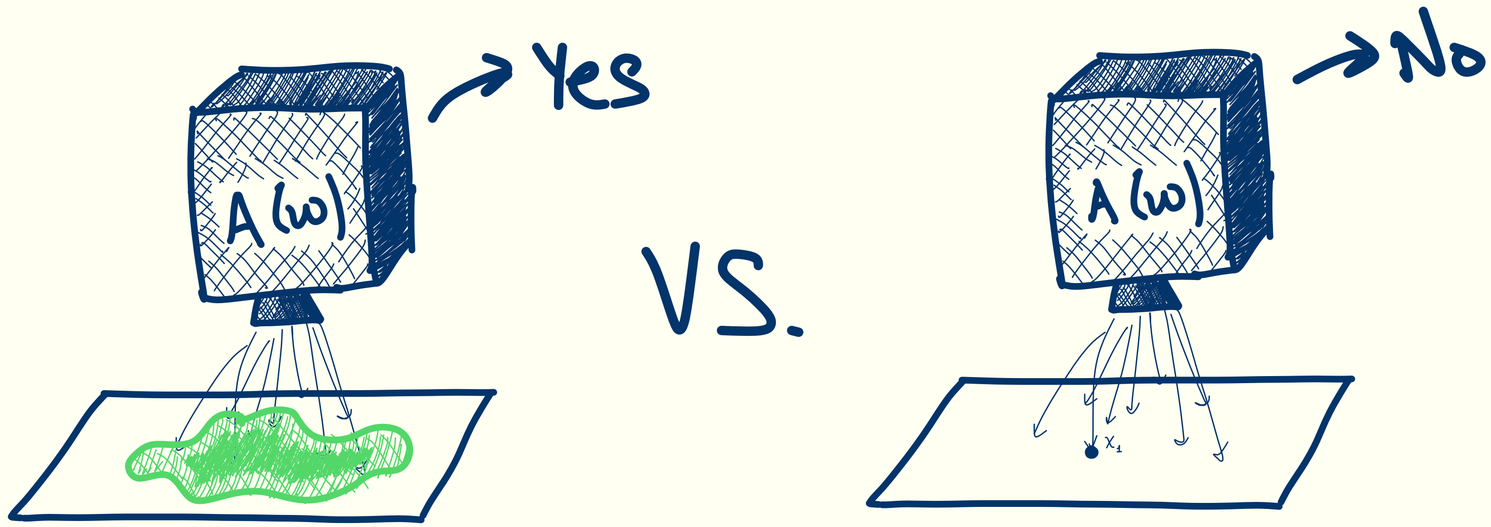

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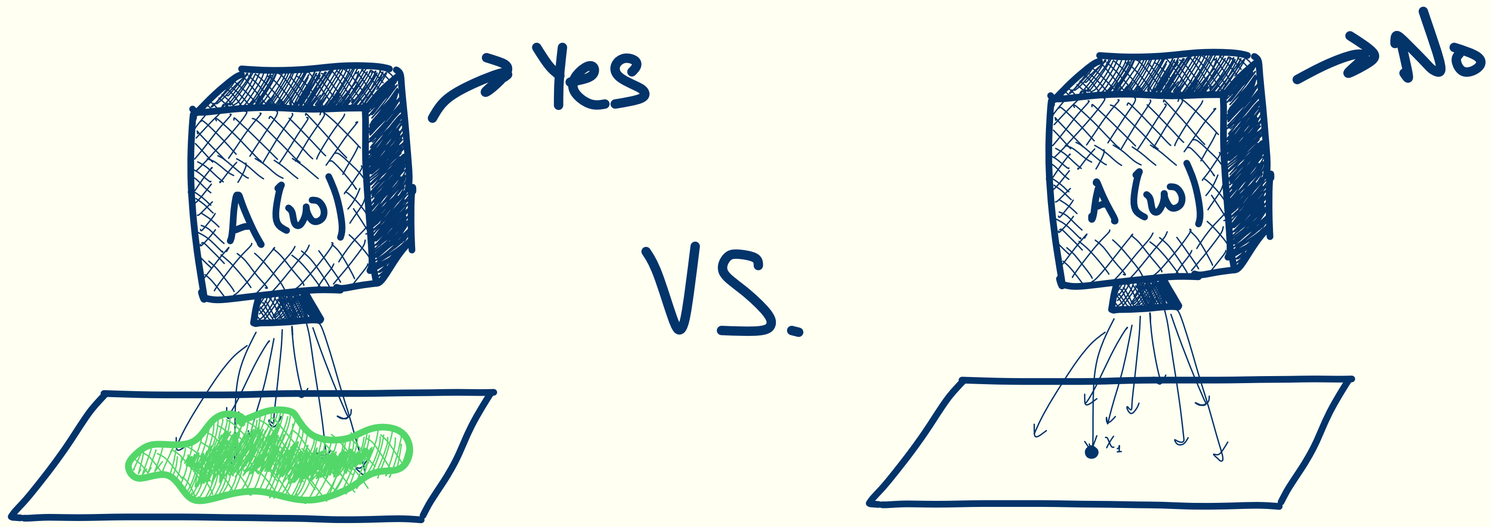

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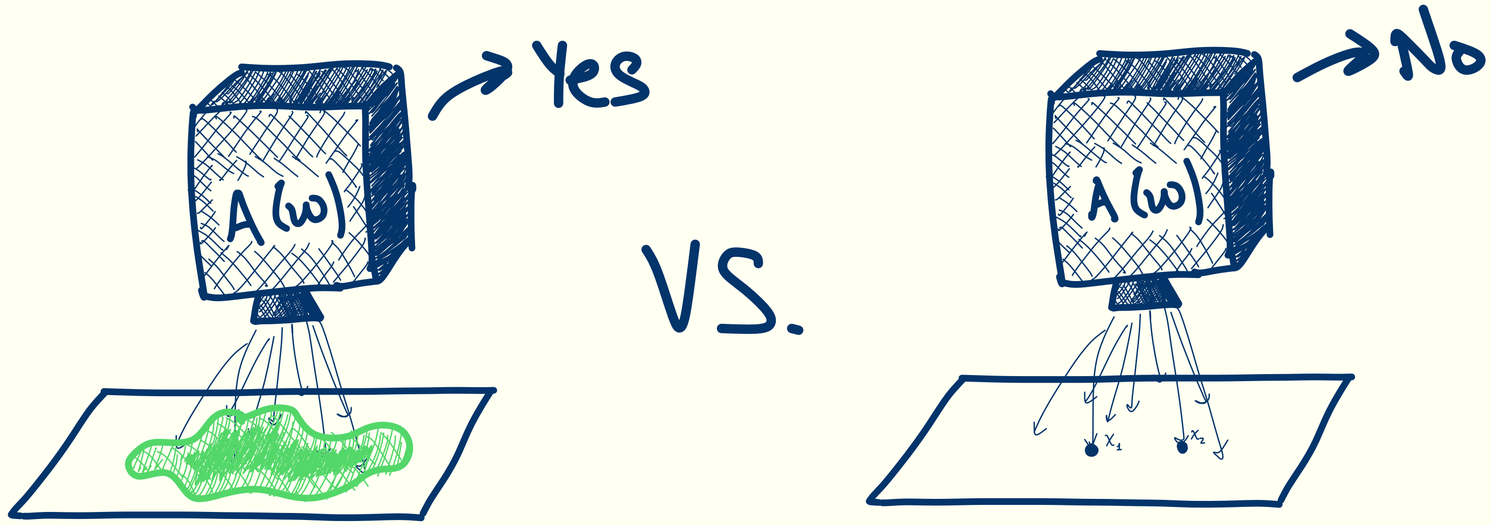
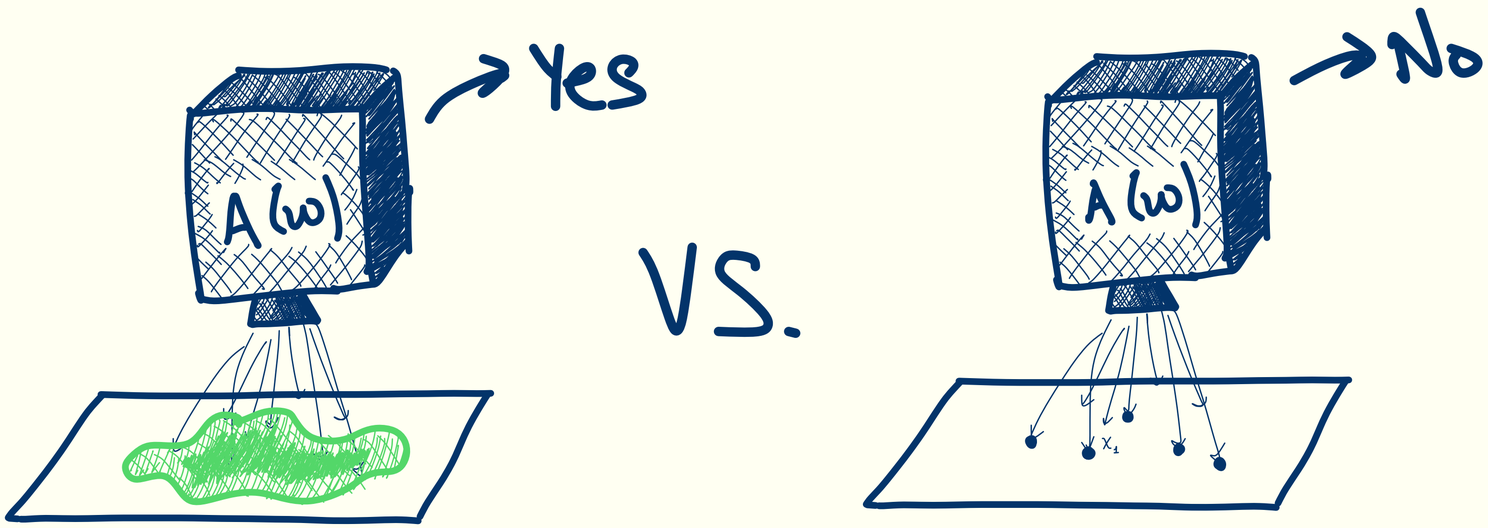


A new approach for ruling out QCMA algorithms

We can keep repeating this to get $v \leq \ell/2$ points from S, with probability at least $(1/36t^2)^v$. If we additionally guess the witness, we can sample v points with probability

$$\geq 2^{-|w|} \left(1/36t^2\right)^v$$



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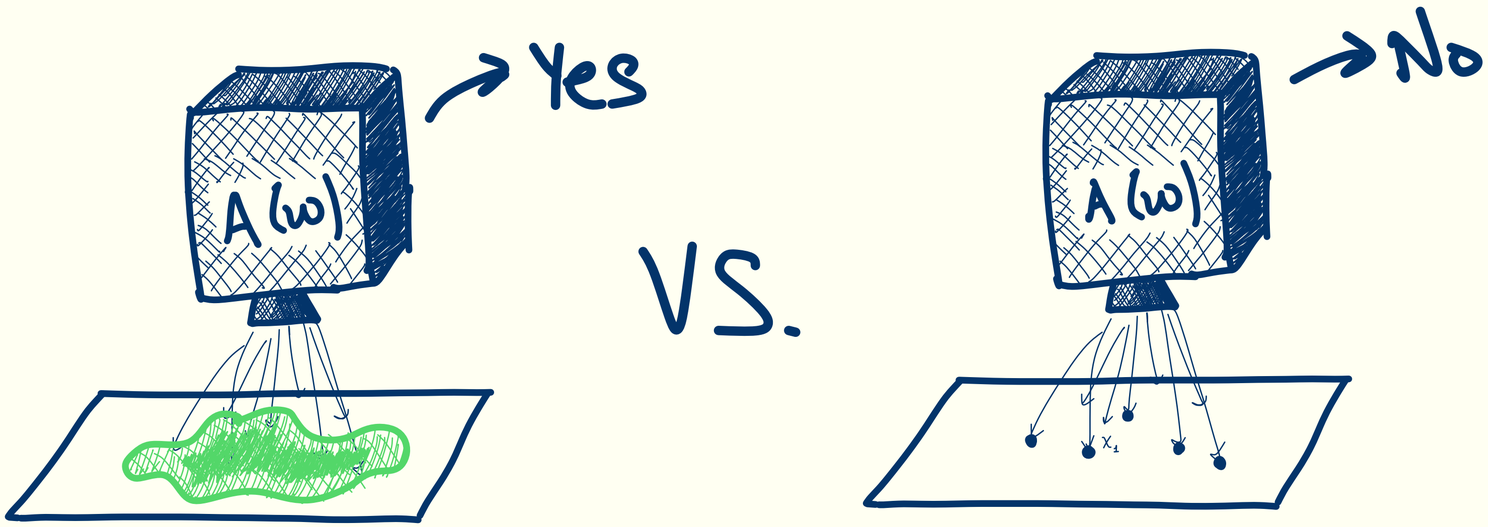
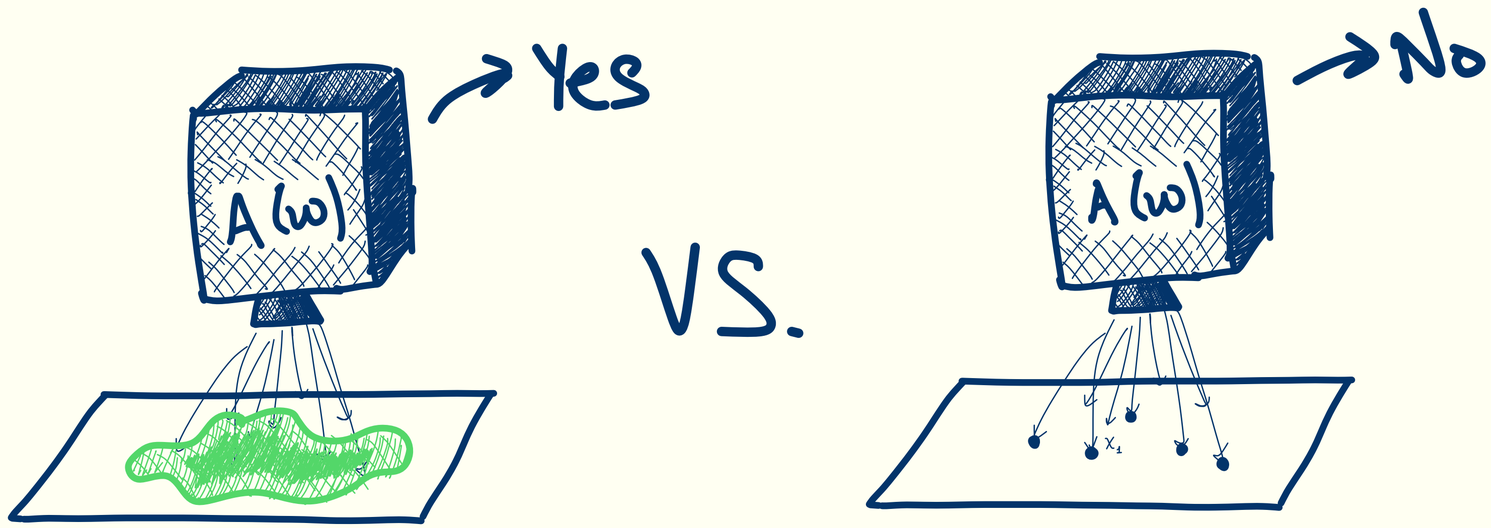


Our sampler doesn't make any queries to S!

A new approach for ruling out QCMA algorithms

If there is a QCMA verifier for set size verification, we get a sampler, making no queries to S, that samples points from any set S with probability $\geq 2^{-|w|}\left(1/36t^2\right)^v$



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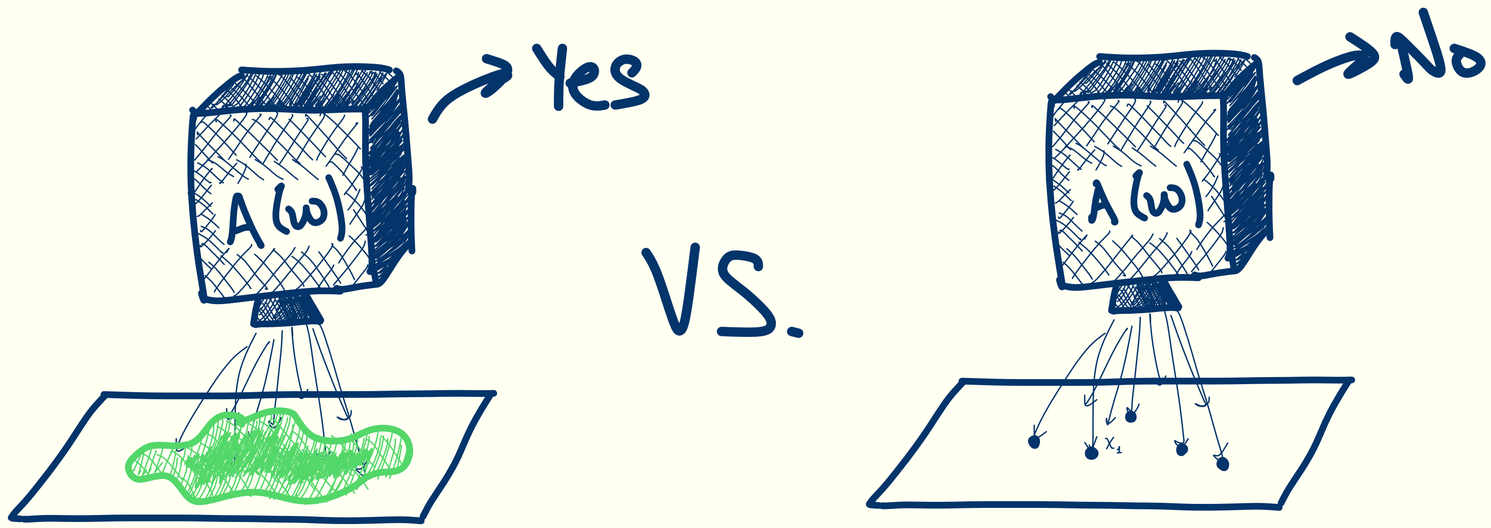

But, any algorithm outputting v points can only be right with probability at most $(\ell/2^n)^v$.
$\rightarrow$ When $v \sim O(|w|/n)$, we get a contradiction!

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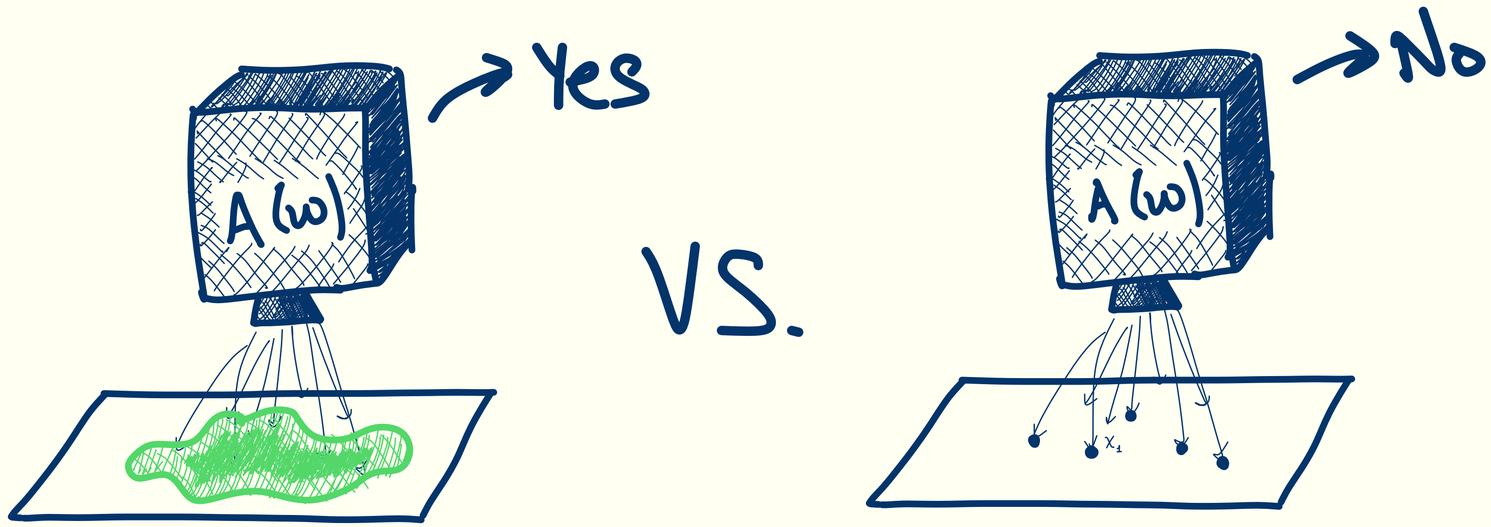
An important thing to note about this proof: It depends on the fact that *w* is "clonable", or that we can re-use it between different rounds. This would not necessarily work for a QMA witness!



From AM to QMA

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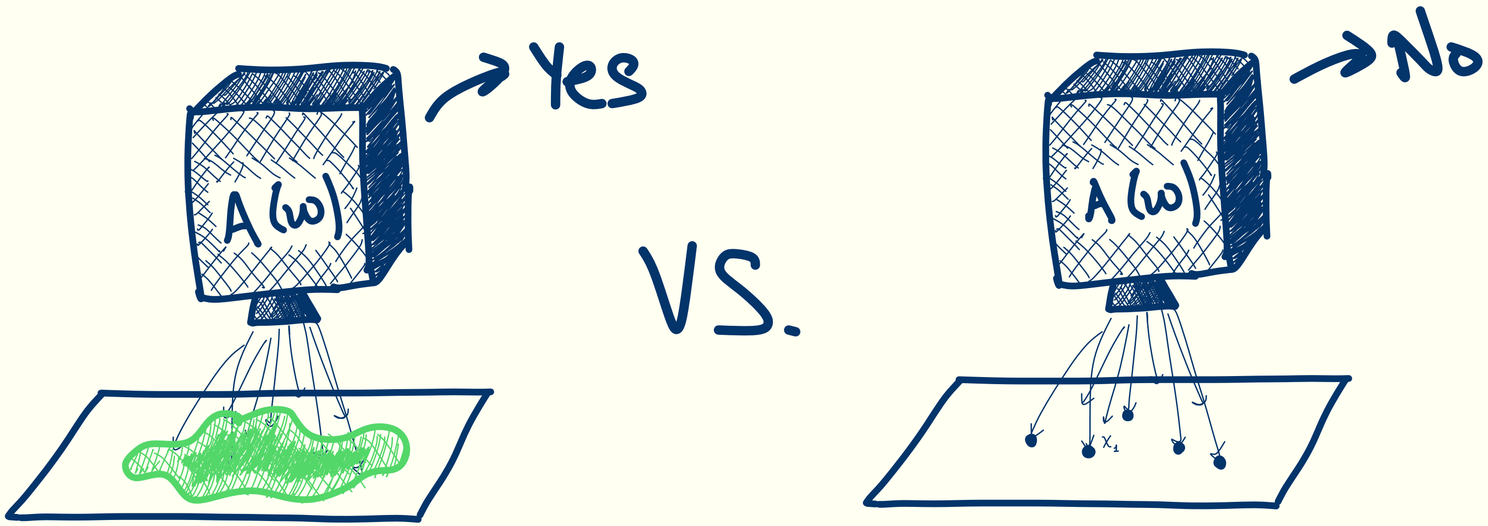

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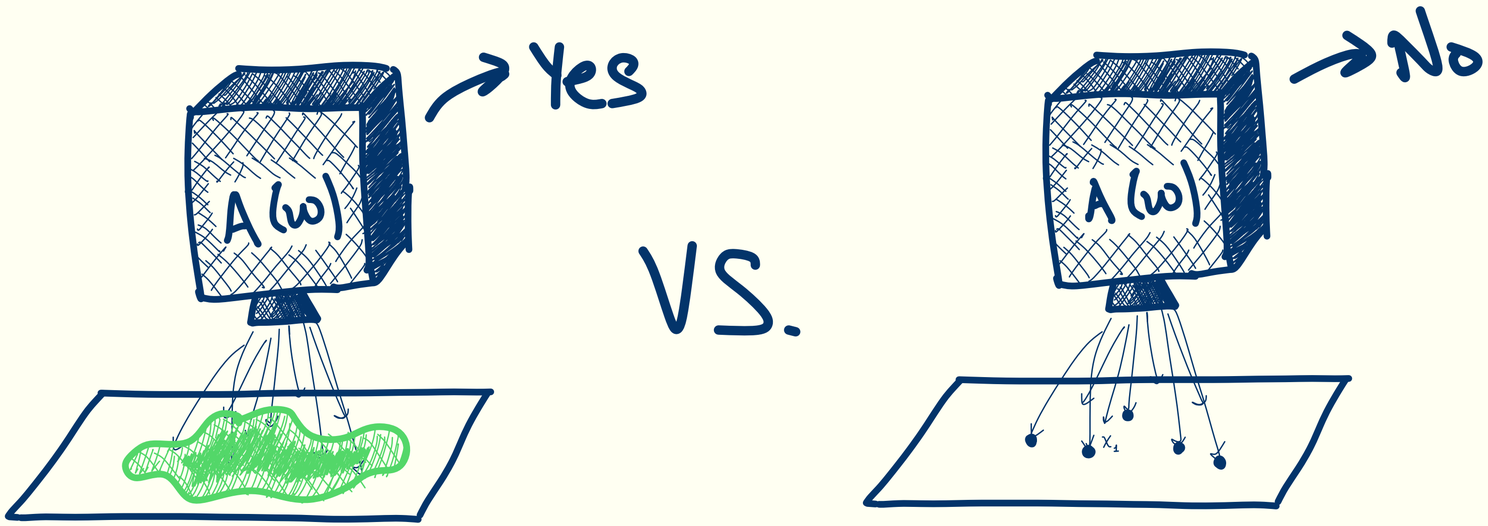

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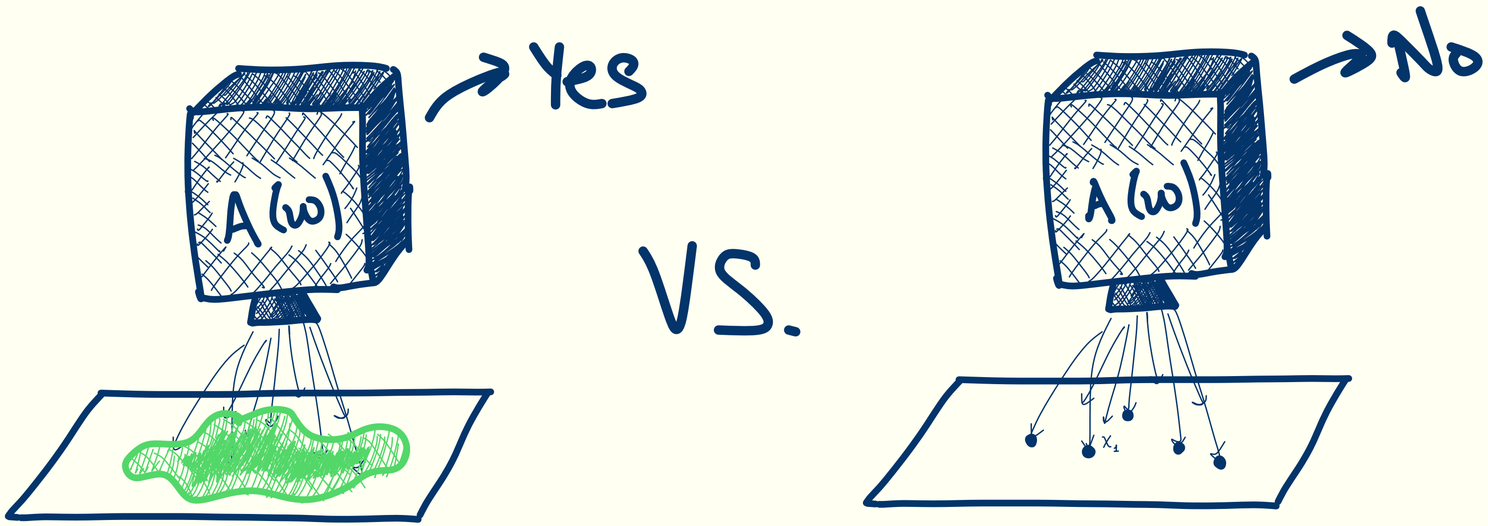
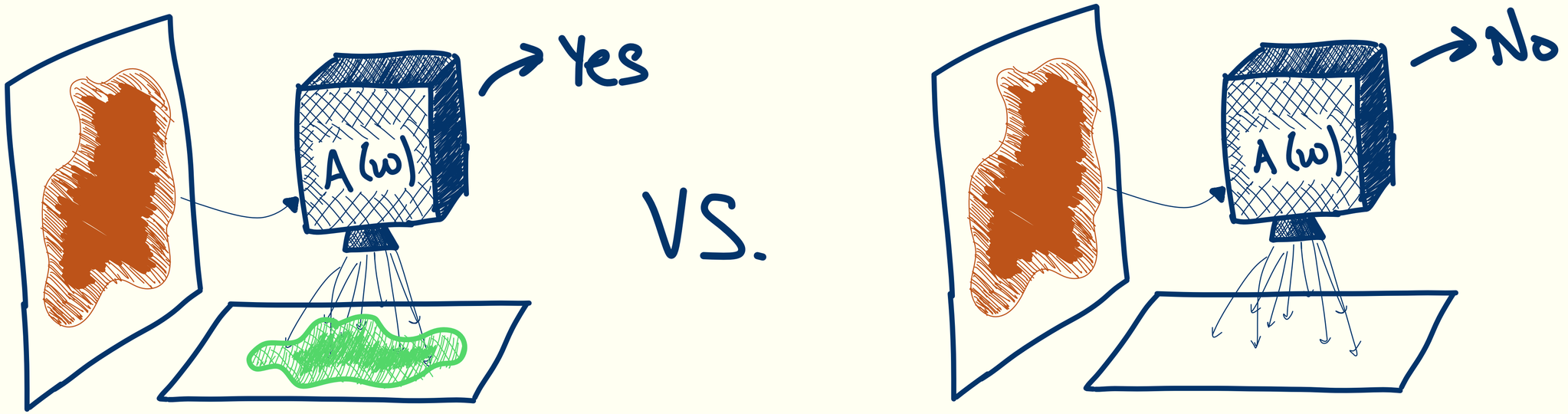
Let's look at one way of putting the problem back in QMA.



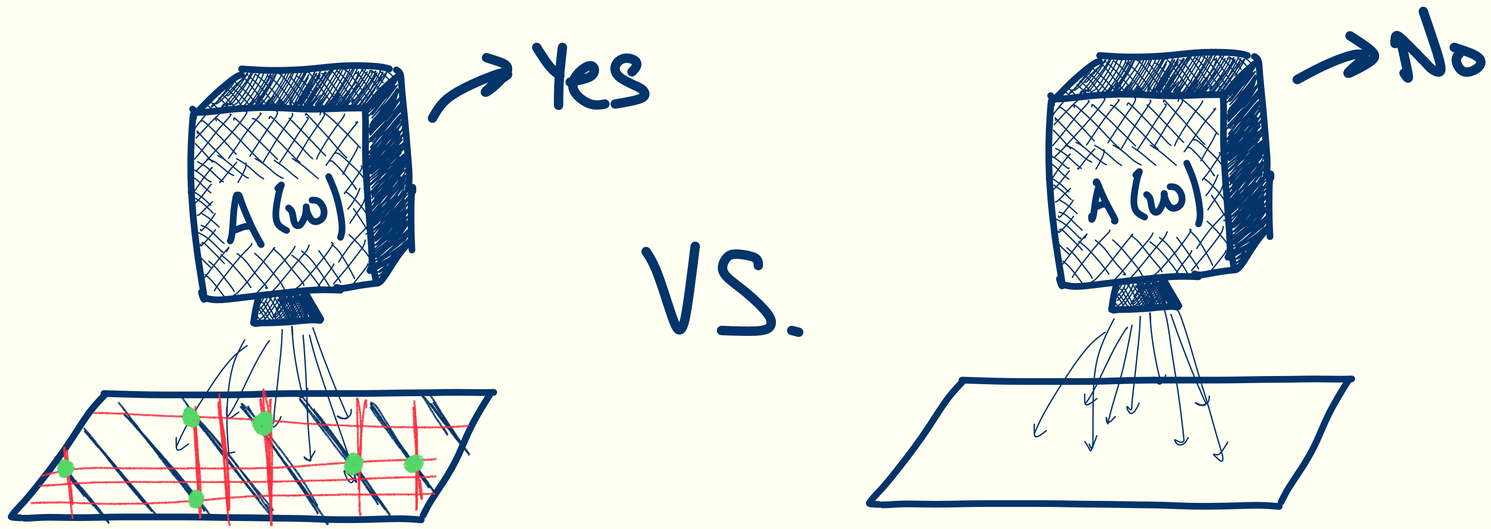
Idea 1: A second oracle [Zhandry'24, BHNZ'25]

Maybe we can add a second oracle that encodes information about the "Fourier transform" of the set S. This results in a problem we call spectral Forrelation.



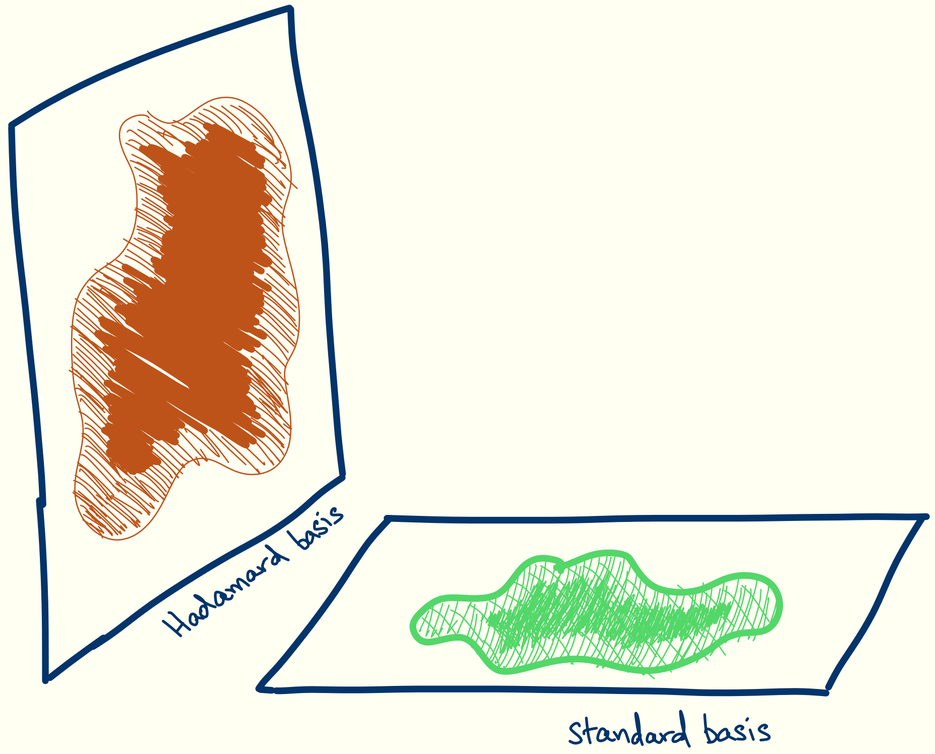
Idea 2: Code intersection [LLPY'23, BK'24, BHV'25]

Another approach will be to add structure to the sparse set S so that a quantum proof can encode many points in the set, but a classical proof can't. This results in a problem called code intersection.



The spectral Forrelation problem

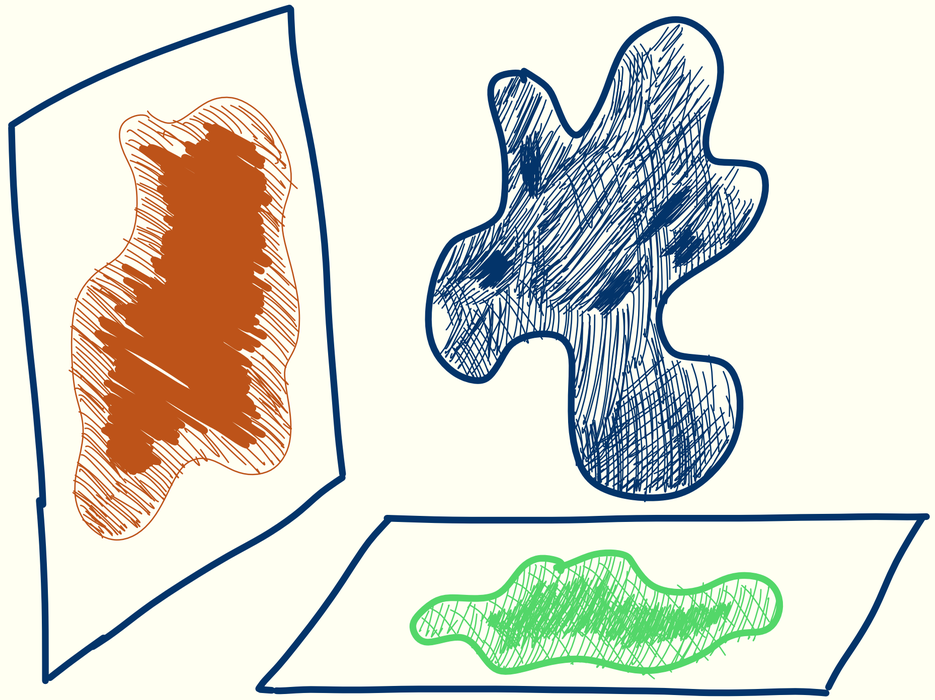
The spectral Forrelation problem is a problem about pairs of sets (S, U), which we treat as oracles through the set membership functions.



The spectral Forrelation problem

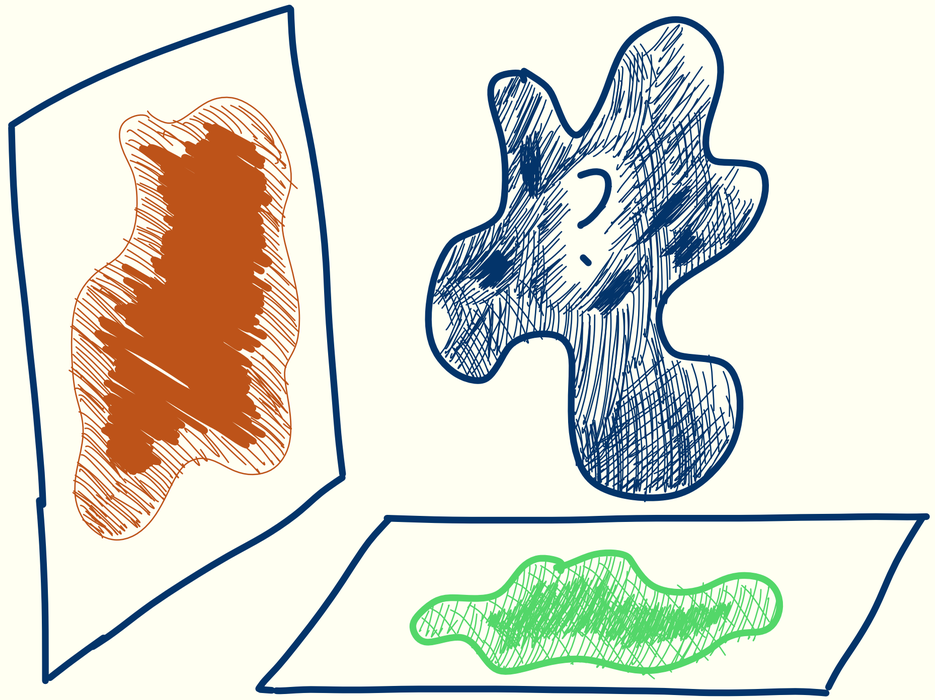
We say that two sets (S, U) are α-spectrally Forrelated if there is a state $|\psi\rangle$ such that

$$\|\Pi_U \cdot H^{\otimes n} \cdot \Pi_S |\psi\rangle\|^2 \geq \alpha$$



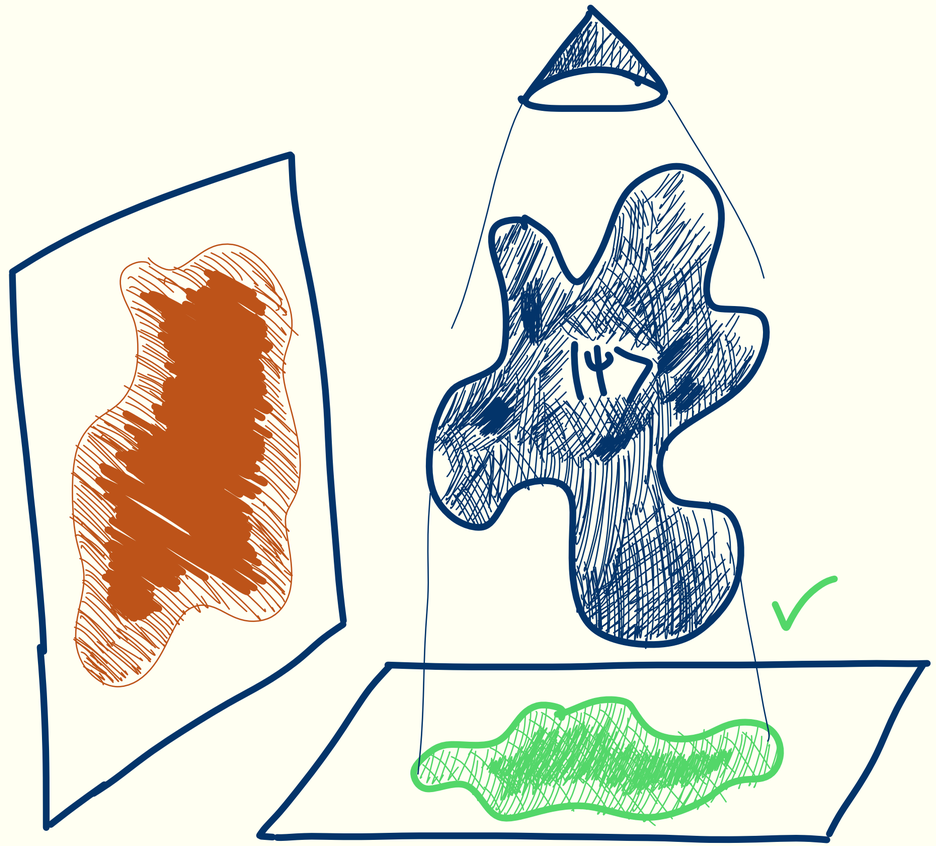
The spectral Forrelation problem

Given oracle access to two sets (S, U) (set membership functions), determine if there is a state $|\psi\rangle$ such that $\|\Pi_U \cdot H^{\otimes n} \cdot \Pi_S |\psi\rangle\|^2$ is large ($\geq 59/100$) or small ($\leq 57/100$), promised that one of the two is the case.



Spectral Forrelation is in QMA

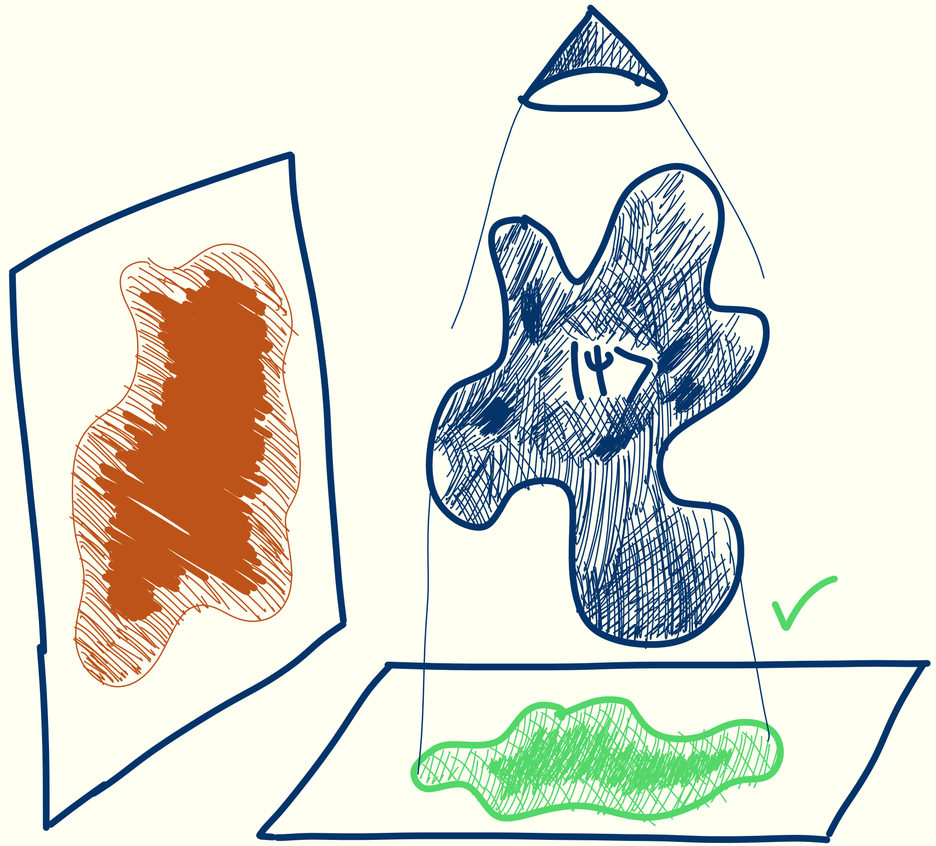
Given a copy of a state $|\psi\rangle$:



Spectral Forrelation is in QMA

Given a copy of a state $|\psi\rangle$:

- Use S oracle to measure the POVM $\{\Pi_S, \mathrm{id} - \Pi_S\}$, reject if the outcome is $\mathrm{id} - \Pi_S$.



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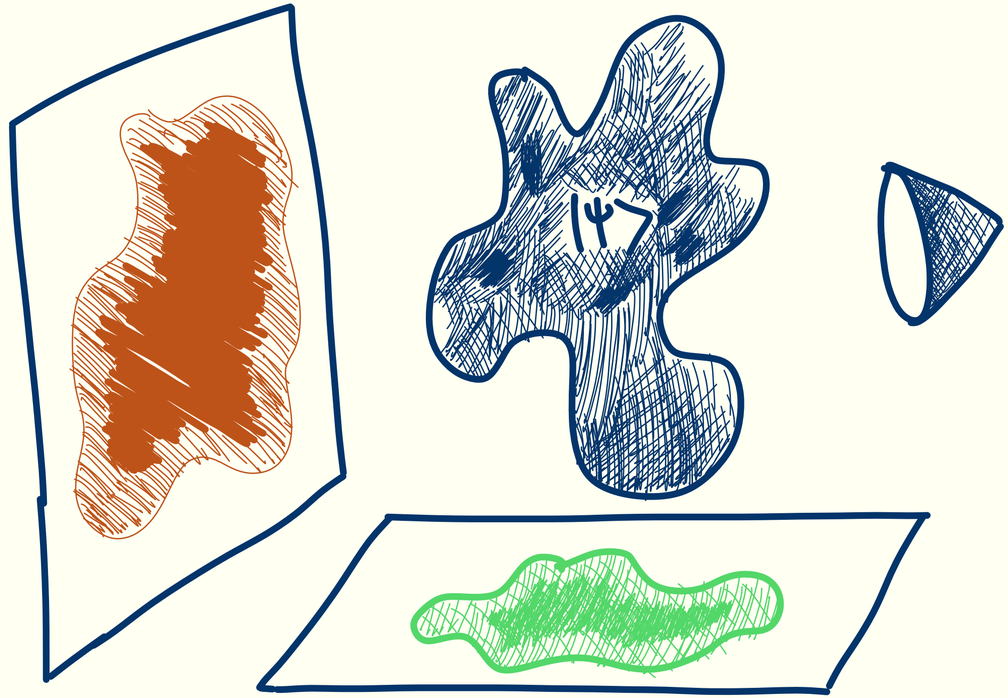

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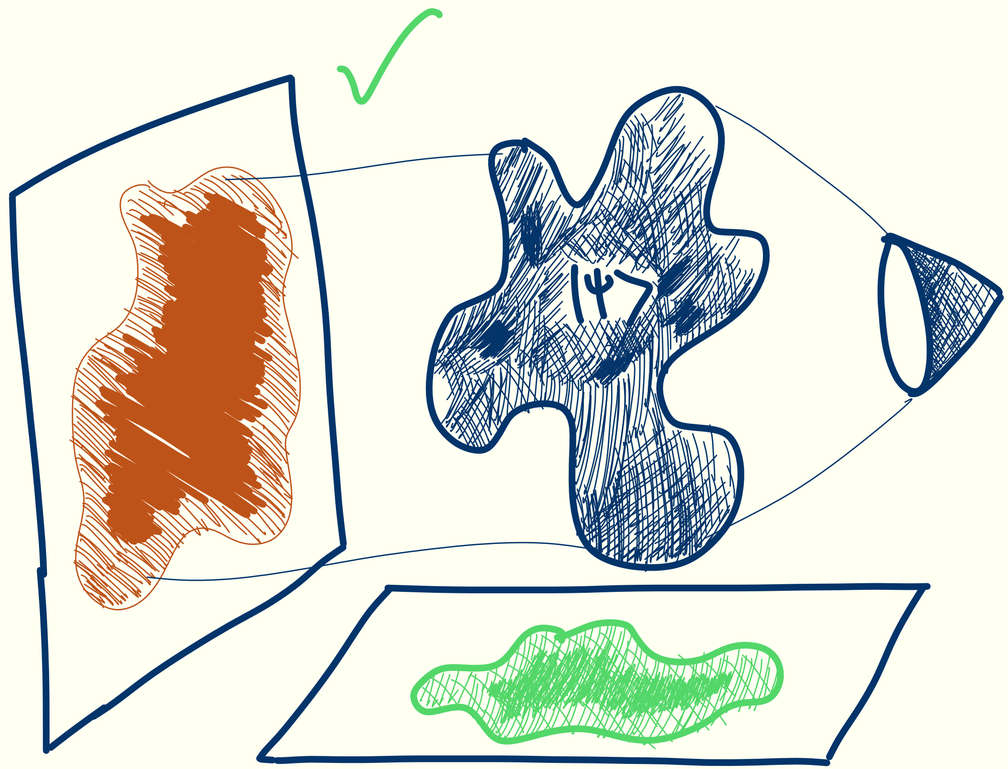
- Accept.



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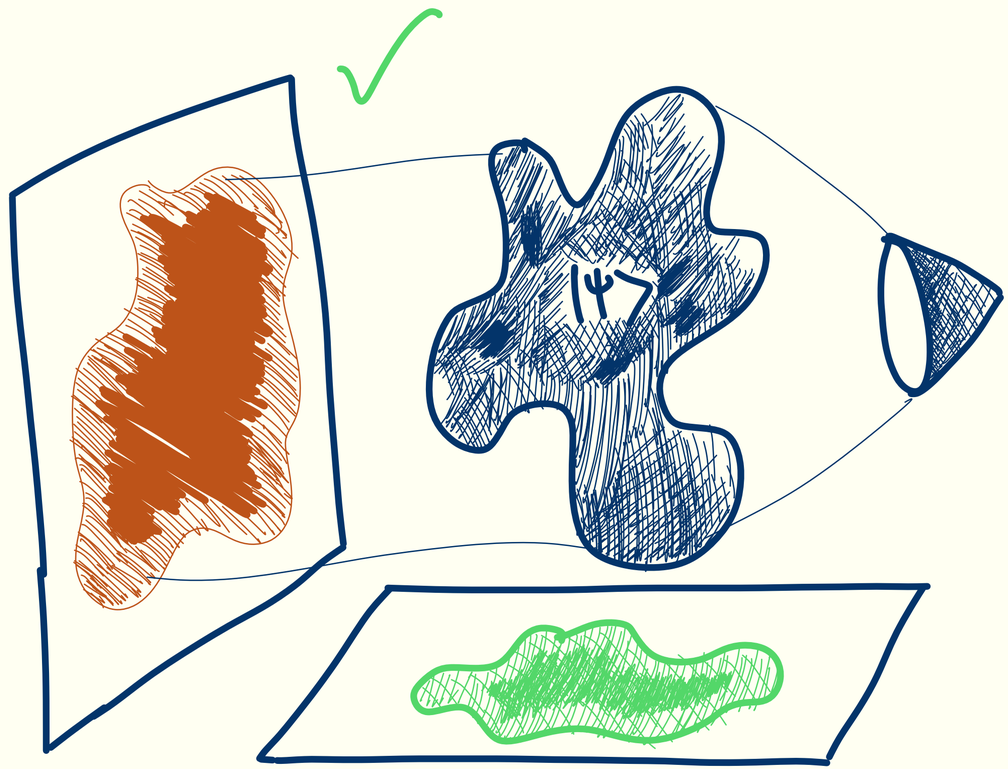
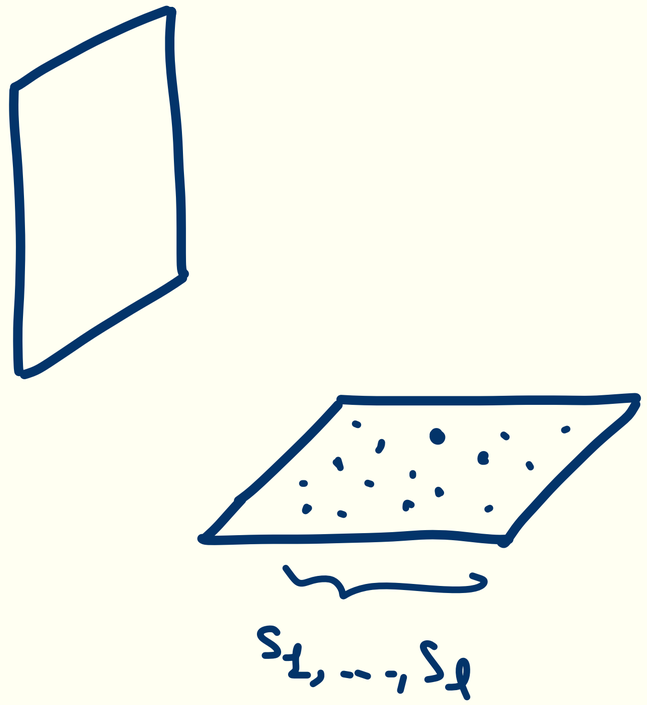


This verifier accepts with probability:
$\|\Pi_U \cdot H^{\otimes n} \cdot \Pi_S |\psi\rangle\|^2$.

Marriot-Watrous amplification can bring this to the standard 2/3 or 1/3.

A distribution over spectral Forrelation yes instances

- We will first sample $\ell = 2^{n/10}$ many random elements $s_1, \ldots, s_\ell$.



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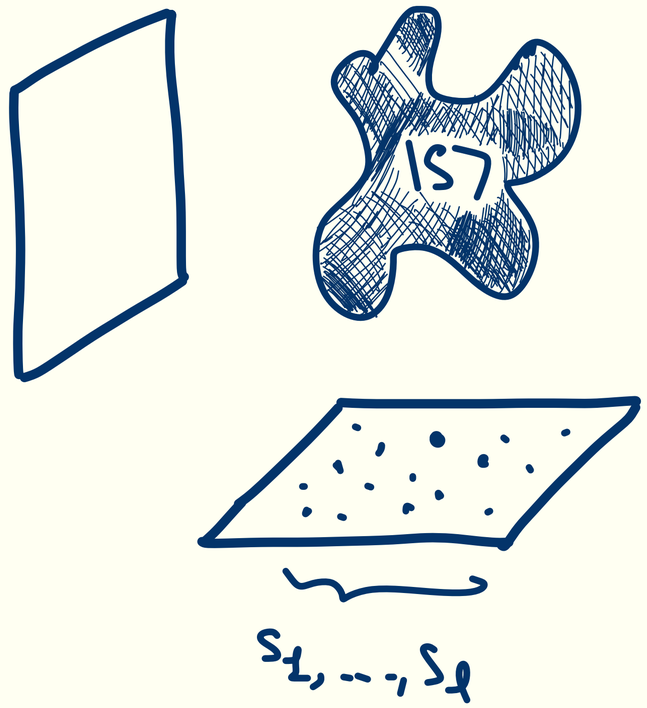

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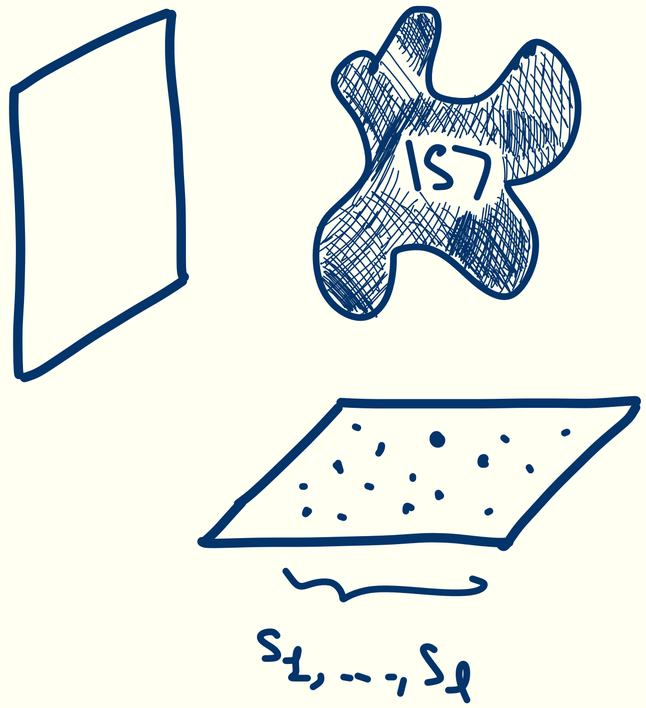
$$\Pr[y \in U] = 1 - \frac{1}{2}\exp\left(-\frac{1}{10}2^n \left|\langle y|H^{\otimes n}|S\rangle\right|^2\right)$$



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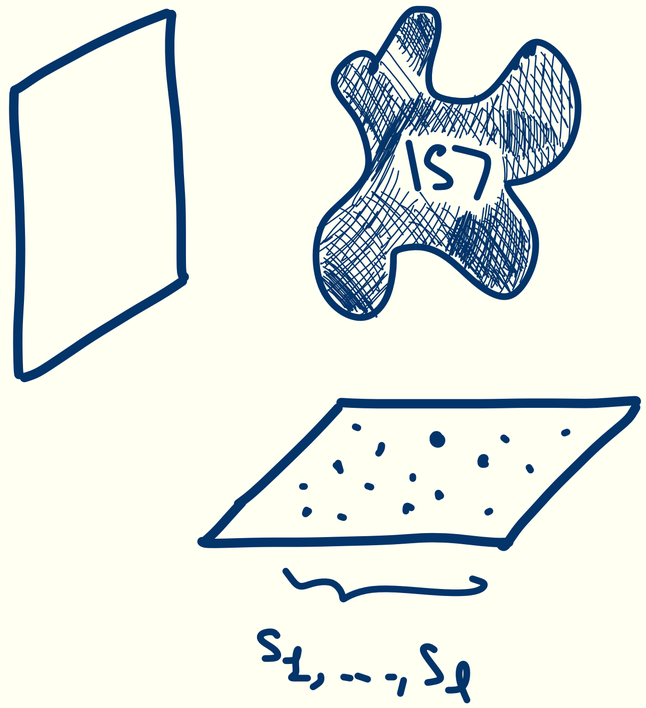

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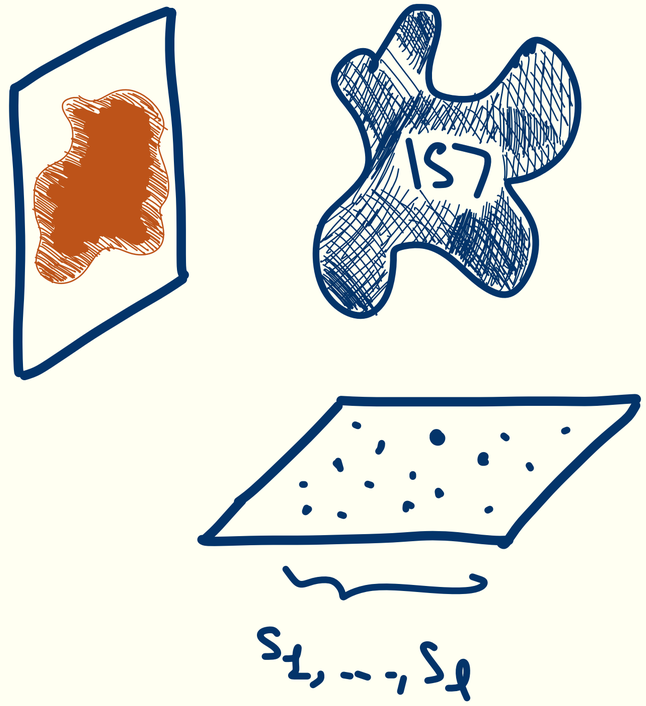

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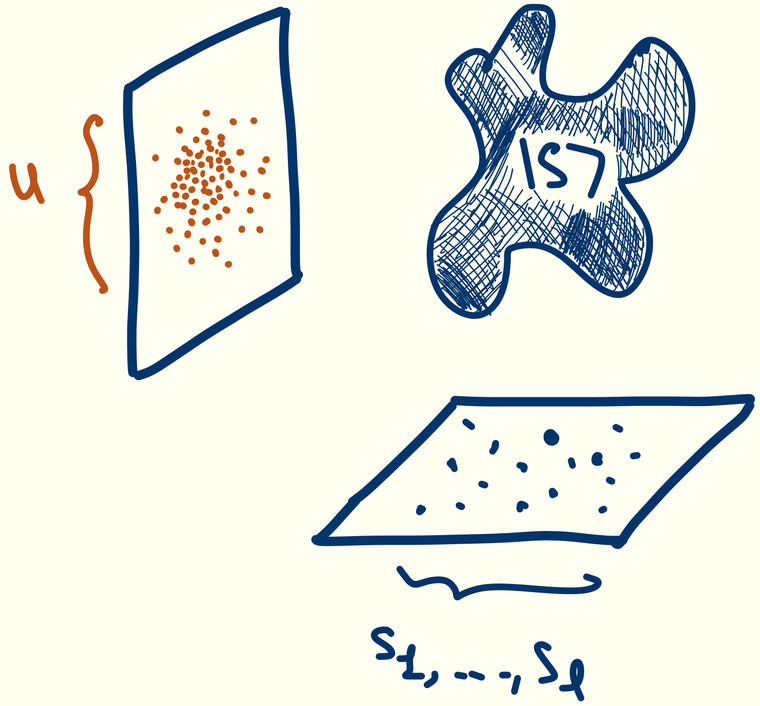

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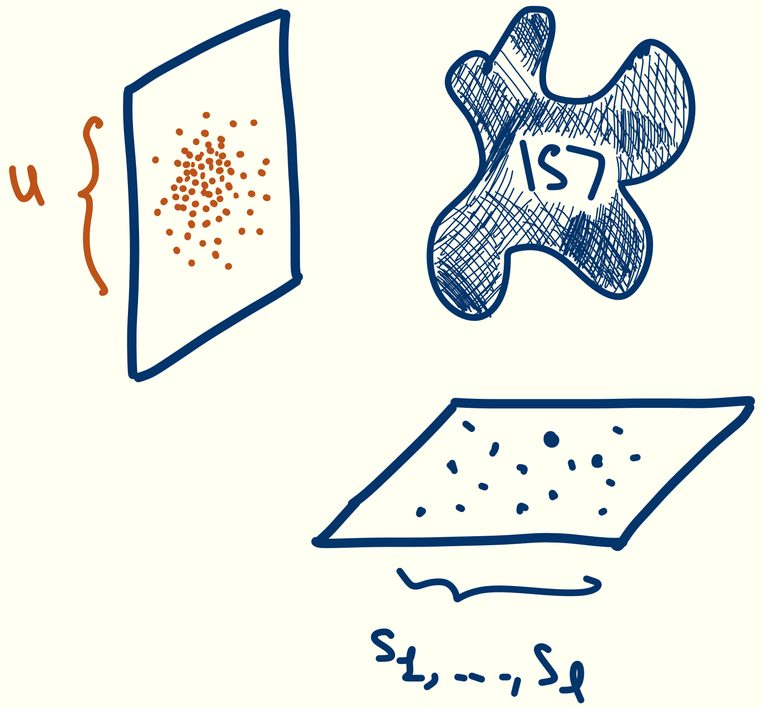
We call this distribution over oracles the Strong distribution.



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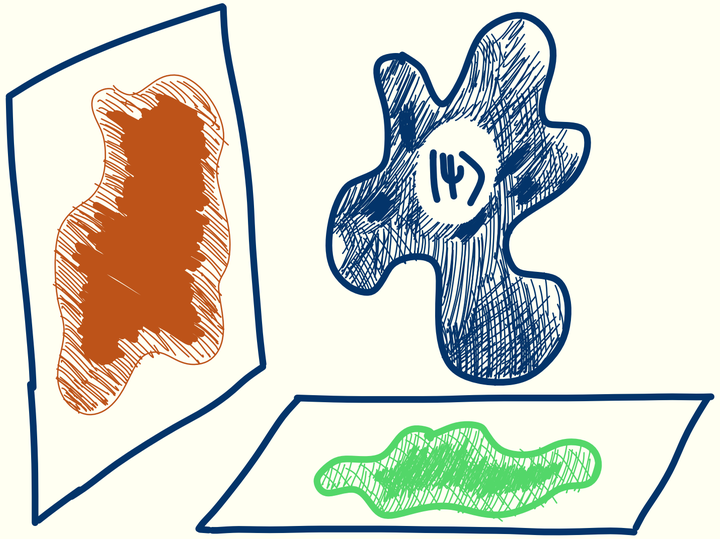

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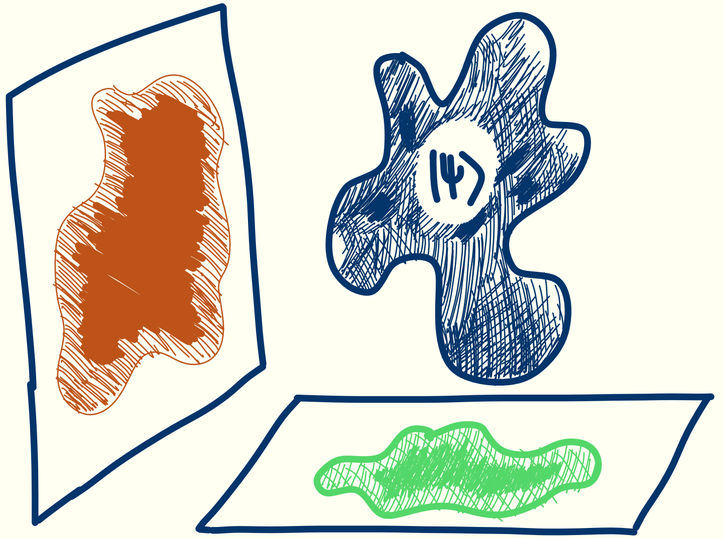
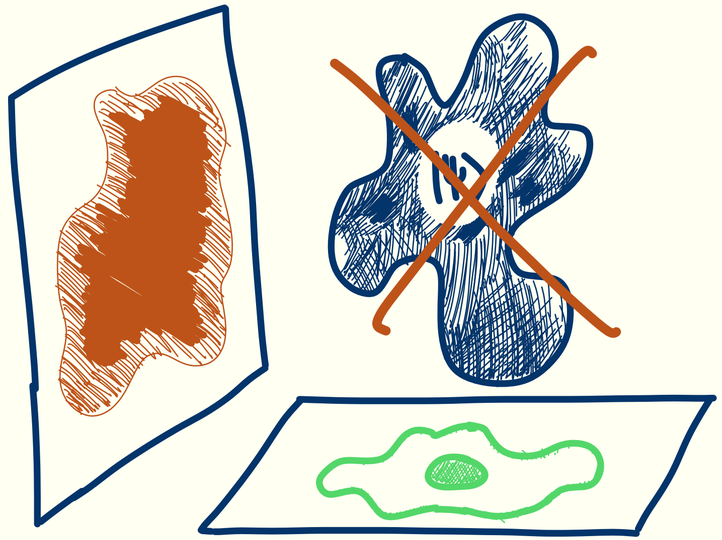
- (S, U) is a yes instance of spectral Forrelation (i.e., $\geq 59/100$ spectrally Forrelated).
- For all $\Delta \subset S$ with $|\Delta| \leq \ell/100$, (Δ, U) is a no instance of spectral Forrelation (i.e., $\leq 57/100$ spectrally Forrelated).





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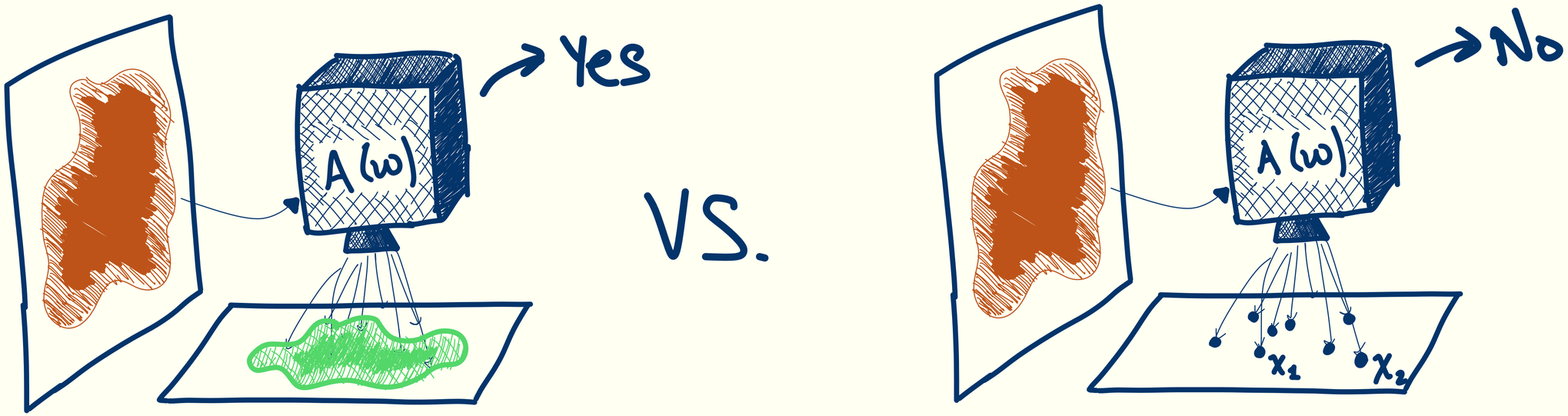
Using concentration bounds, we get that with very high probability, this happens.

Applying the sampler idea to spectral Forrelation

Theorem: If there exists a QCMA algorithm, making t queries to (S, U) and taking a witness of length w, then for all $0 < v < \ell/100$, there is a query algorithm making vt queries to U that outputs v distinct points from S with probability

$$\geq 2^{-w} \left(\frac{1}{36t^2} \right)^v$$



Sampler upper bounds for Strong

The Fock basis is a way to write down a multi-set, similar to how we write subsets of $\{0,1\}^n$ as 2^n bit strings. Given a multi-set with ℓ_x copies of x, we associate it with a vector of 2^n non-negative integers:

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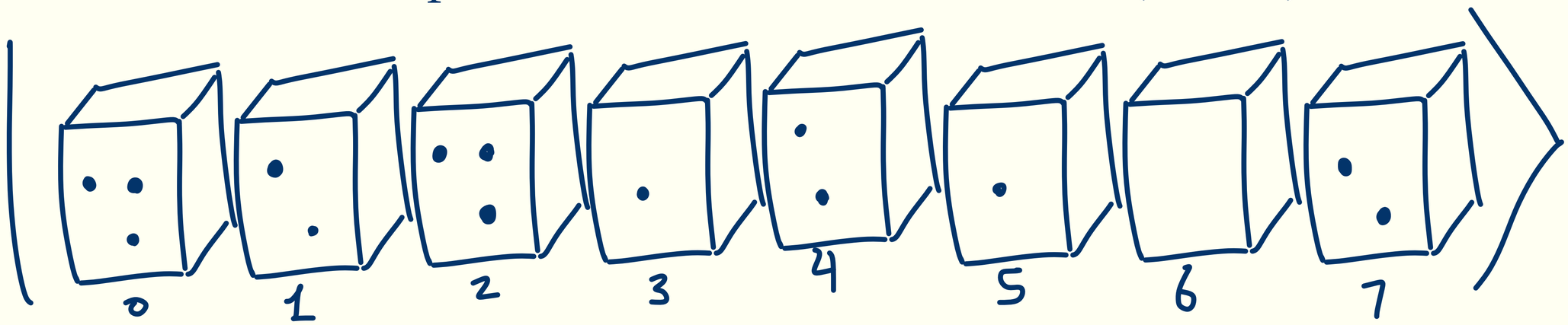
Then the uniform superposition over multi-sets is given by

$$\frac{1}{\sqrt{2^n}} \sum_{\vec{\ell}} \sqrt{\frac{\ell!}{\prod_x \ell_x!}} |\ell_0, \ldots, \ell_{2^n}\rangle .$$

Sampler upper bounds for Strong: Bosons

Bosons are a mathematical representation of multi-sets used in physics. The "annihilation" and "creation" operators add and subtract elements (bosons).



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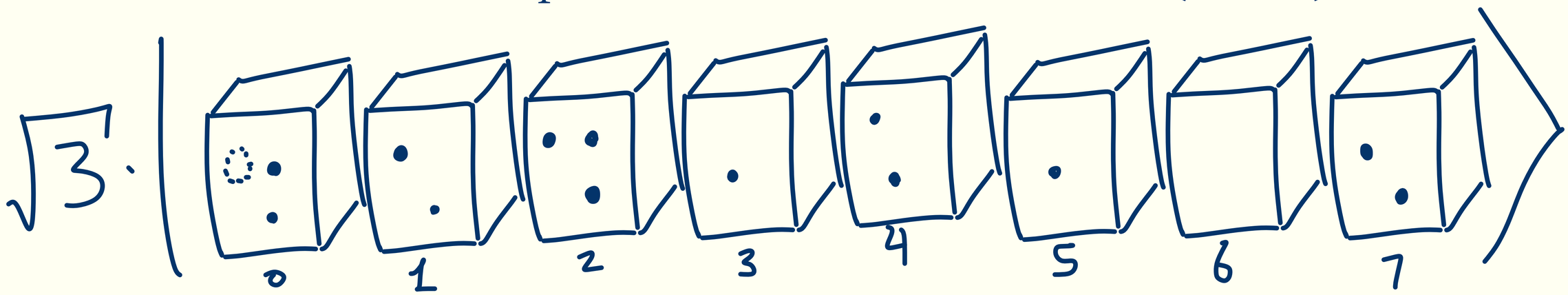

$$\hat{a}_x \,|\, \ell_0, \ldots, \ell_x, \ldots, \ell_{2^n-1}\rangle = \sqrt{\ell_x}\,|\, \ell_0, \ldots, \ell_x - 1, \ldots, \ell_{2^n-1}\rangle$$

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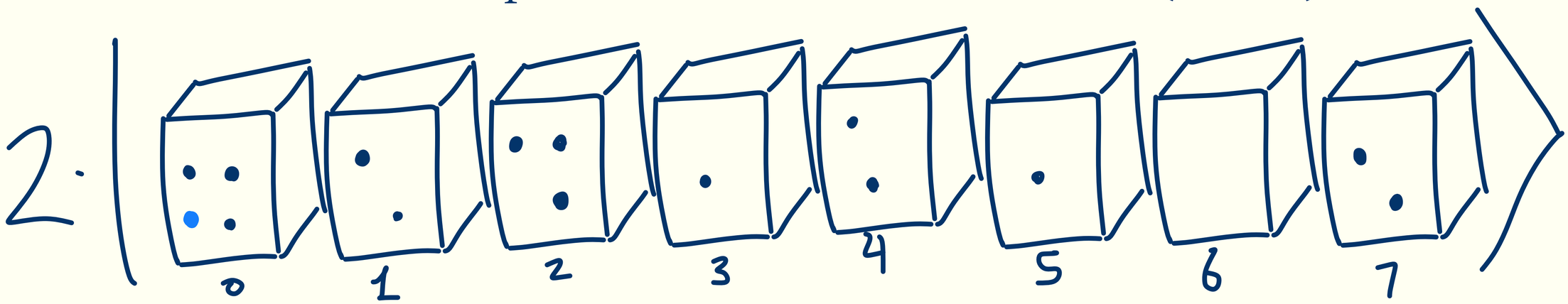

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$$\hat{a}_x^\dagger \,|\, \ell_0, \ldots, \ell_x, \ldots, \ell_{2^n-1} \rangle = \sqrt{\ell_x + 1} \,|\, \ell_0, \ldots, \ell_x + 1, \ldots, \ell_{2^n-1} \rangle$$

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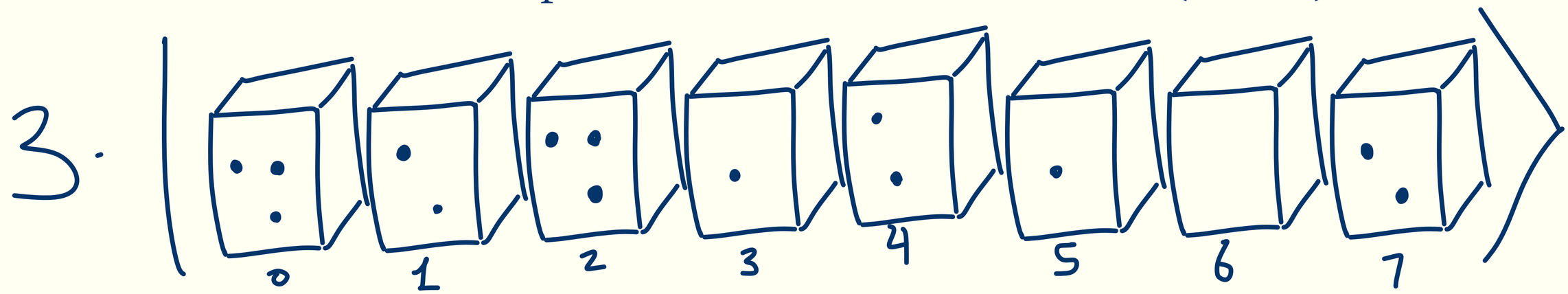

$$\hat{a}_x \left| \ell_0, \ldots, \ell_x, \ldots, \ell_{2^n-1} \right\rangle = \sqrt{\ell_x} \left| \ell_0, \ldots, \ell_x - 1, \ldots, \ell_{2^n-1} \right\rangle$$

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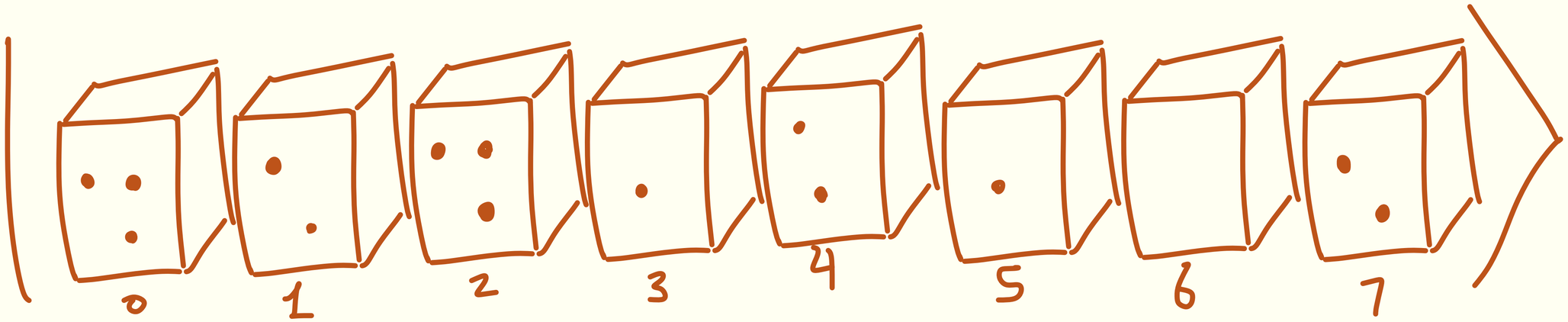
(after $\hat{n}_0$)

Taking the product, the number operator $\hat{n}_x = \hat{a}_x^\dagger \hat{a}_x$ is diagonal in the position Fock basis, and applies a scaling of the number of bosons in the x'th mode.

Sampler upper bounds for Strong: Bosons

We can also define a “Hadamard” basis for the bosons, with the analogous operators:

$$\tilde{a}_y = \frac{1}{\sqrt{2^n}} \sum_x (-1)^{y \cdot x} \hat{a}_x \quad \text{and} \quad \tilde{a}_y^\dagger = \frac{1}{\sqrt{2^n}} \sum_x (-1)^{y \cdot x} \hat{a}_x^\dagger$$

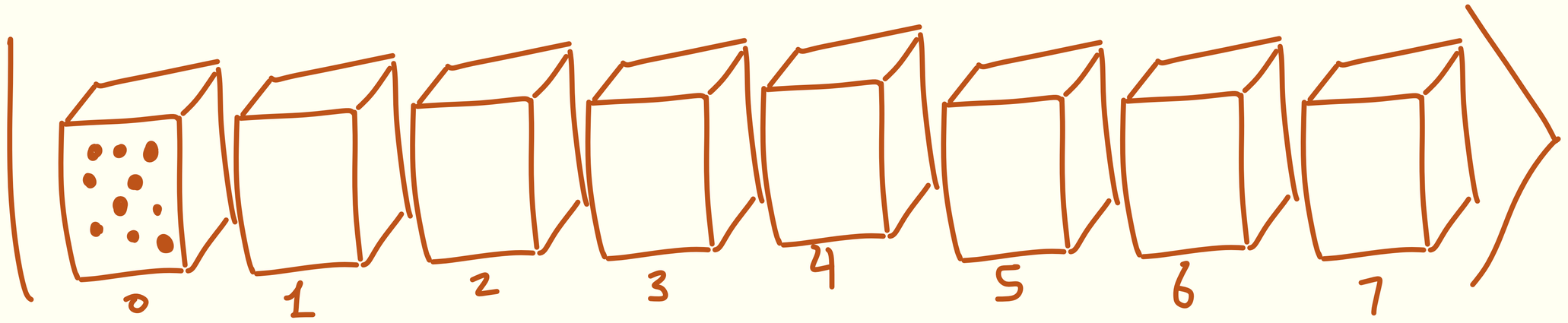


As a matter of notation, we also define $|\text{vac}\rangle$ to be the state $|0,\ldots,0\rangle$.

Sampler upper bounds for Strong: Bosons

Claim: The purification of a random multi-set is:

$$|\,\text{init}\rangle = \frac{1}{\sqrt{\ell\,!}} \left(\tilde{a}_0^\dagger\right)^\ell |\,\text{vac}\rangle$$



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Proof: Expand out the expression:

$$\frac{1}{\sqrt{\ell\,!}}\left(\tilde{a}_0^\dagger\right)^\ell |\,\text{vac}\rangle = \frac{1}{\sqrt{2^{n\ell}\cdot\ell\,!}} \sum_{s_1,\ldots,s_\ell} \hat{a}_{s_1}^\dagger \ldots \hat{a}_{s_\ell}^\dagger |\,\text{vac}\rangle$$

If you deal with the coefficients (and multiplicities of the multi-sets), it works out. 😊

Compressed oracles for Strong: Action of U

Roughly, querying U at some fixed y is like querying the squared Fourier coefficient $\gamma_y^{(S)}$.
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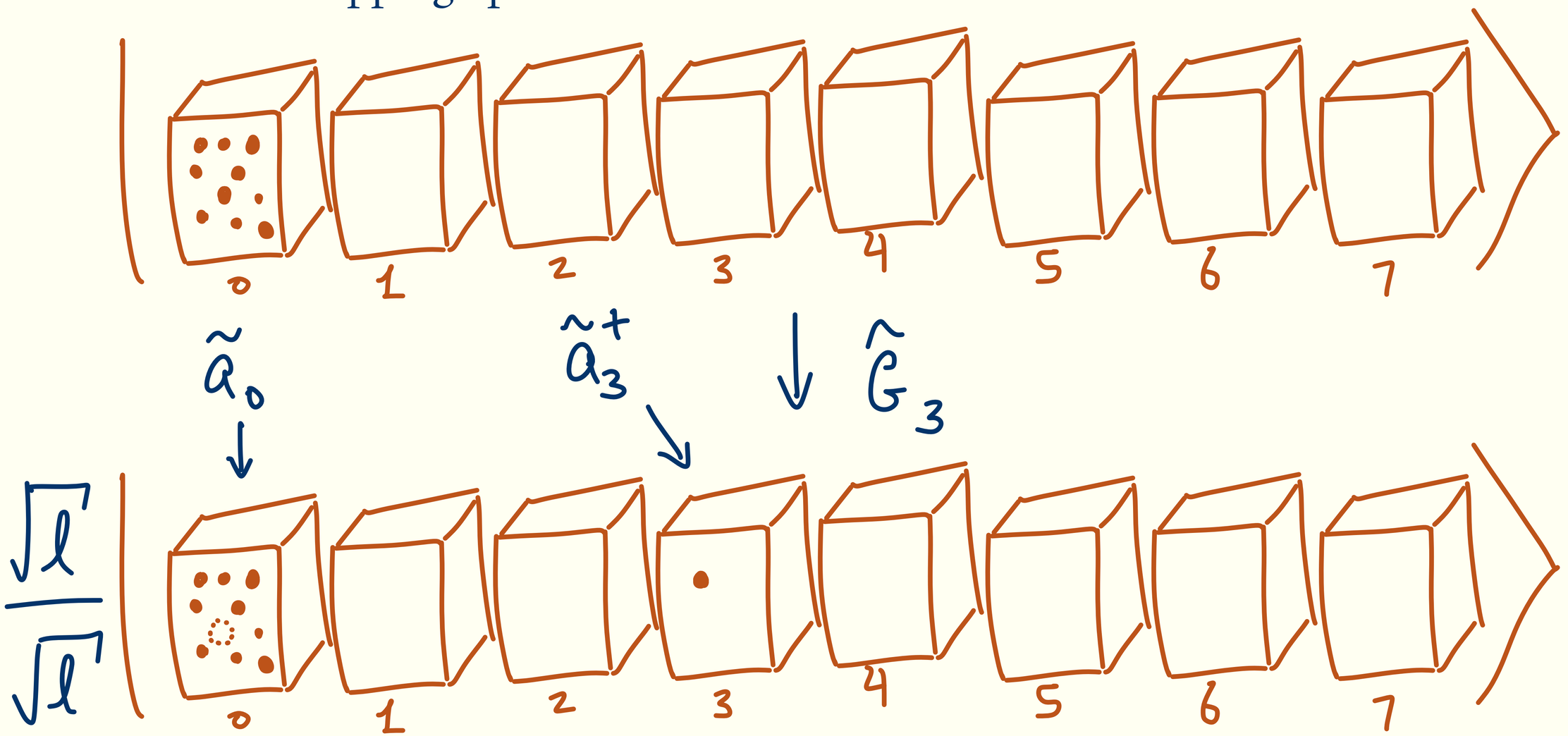
Let's define the momentum hopping and double hopping operators

$$\widetilde{G}_y = \frac{1}{\sqrt{\ell}} \sum_{x \in \{0,1\}^n} \widetilde{a}^\dagger_{x \oplus y} \widetilde{a}_x \quad \text{and} \quad \widetilde{H}_y = \frac{1}{\ell} \sum_{x,x' \in \{0,1\}^n} \widetilde{a}^\dagger_{x \oplus y} \widetilde{a}^\dagger_{x' \oplus y} \widetilde{a}_x \widetilde{a}_{x'}$$

Compressed oracles for Strong: Action of U

What does the hopping operator do.



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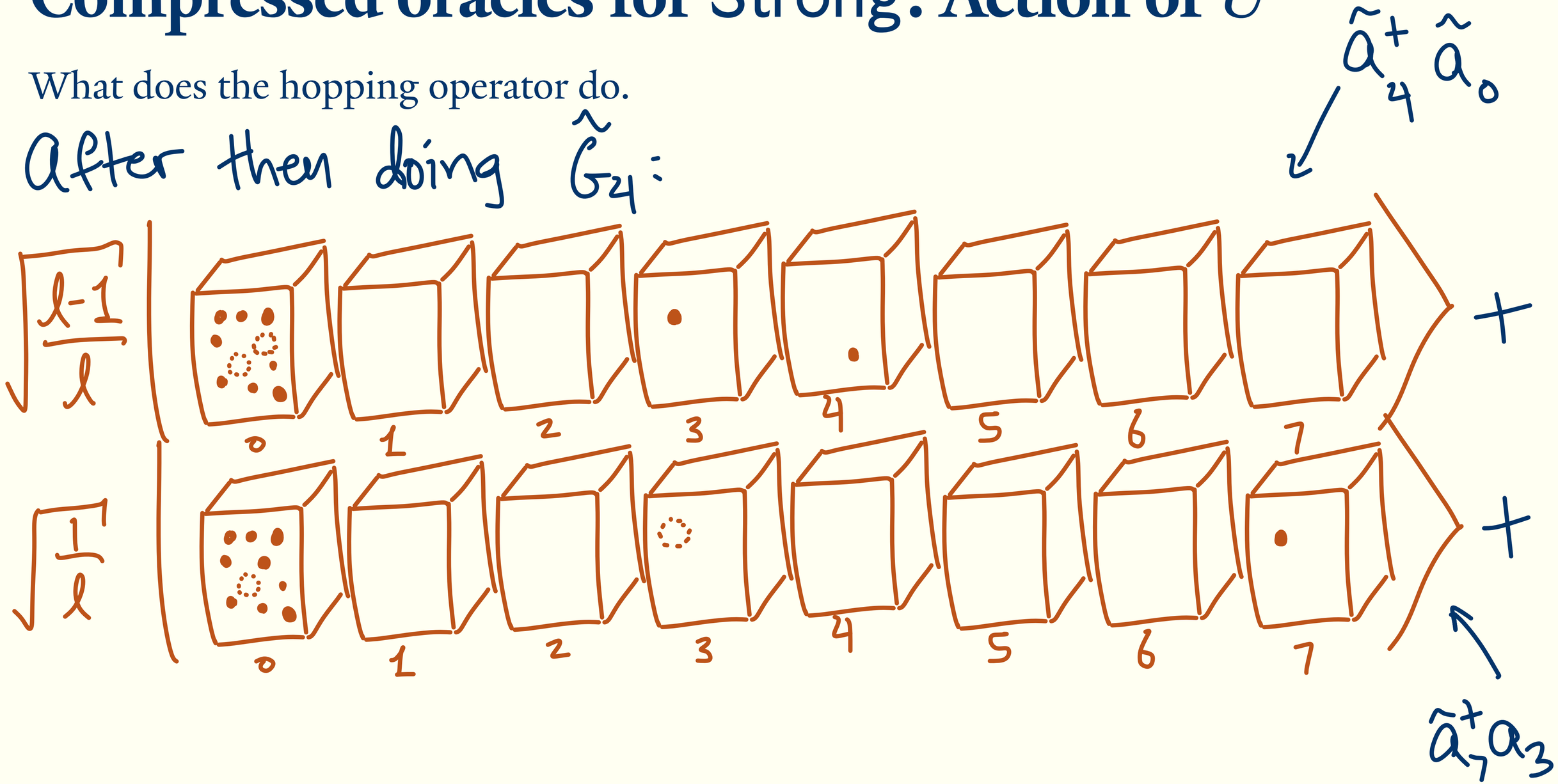

Compressed oracles for Strong: Action of U

Claim: The diagonal matrix that applies the squared Fourier coefficient is actually:

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Proof: We can expand out a position Fock state in the momentum basis and directly compute the action of the hopping operator (the double hopping is the square):

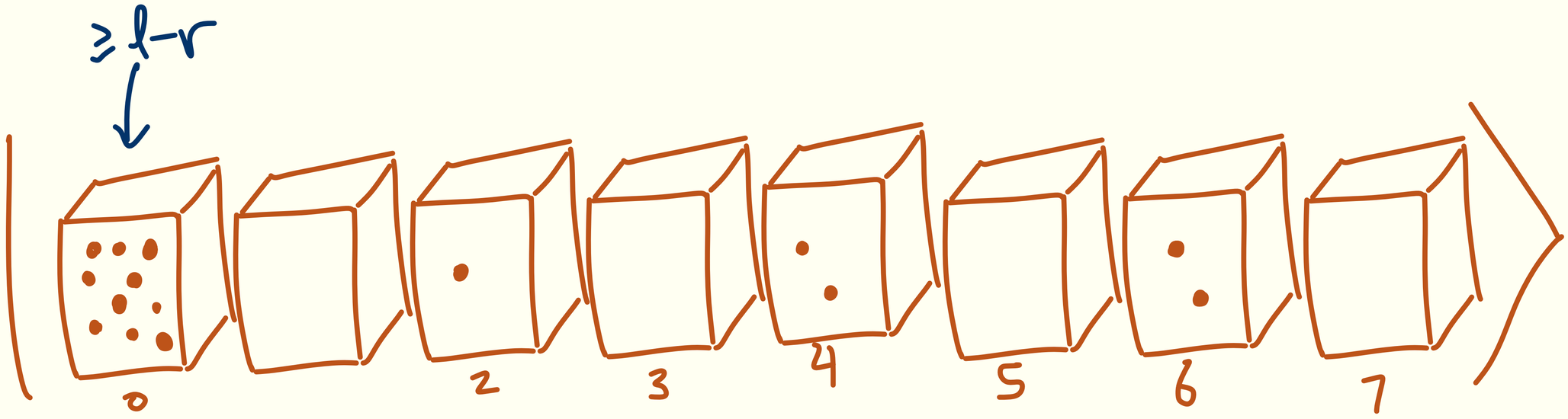
$$\widetilde{G}_y \hat{a}_{s_1}^\dagger \ldots \hat{a}_{s_\ell}^\dagger |\mathrm{vac}\rangle = \widetilde{G}_y \sum_{t_1,\ldots,t_\ell} \left(\prod_i (-1)^{t_i \cdot s_i} \right) \widetilde{a}_{t_1}^\dagger \ldots \widetilde{a}_{t_\ell}^\dagger |\mathrm{vac}\rangle$$

When we apply the hop and re-index the sum, we see that we just get a phase kickback!

Quasi-even condensates

A (r, o)-quasi-even condensate is a momentum Fock state $|\ell_0, \ldots, \ell_{2^n}\rangle$ that satisfies:

Condensate: $\ell_0 \geq \ell - r$, i.e., almost all of the bosons are in their initial position.



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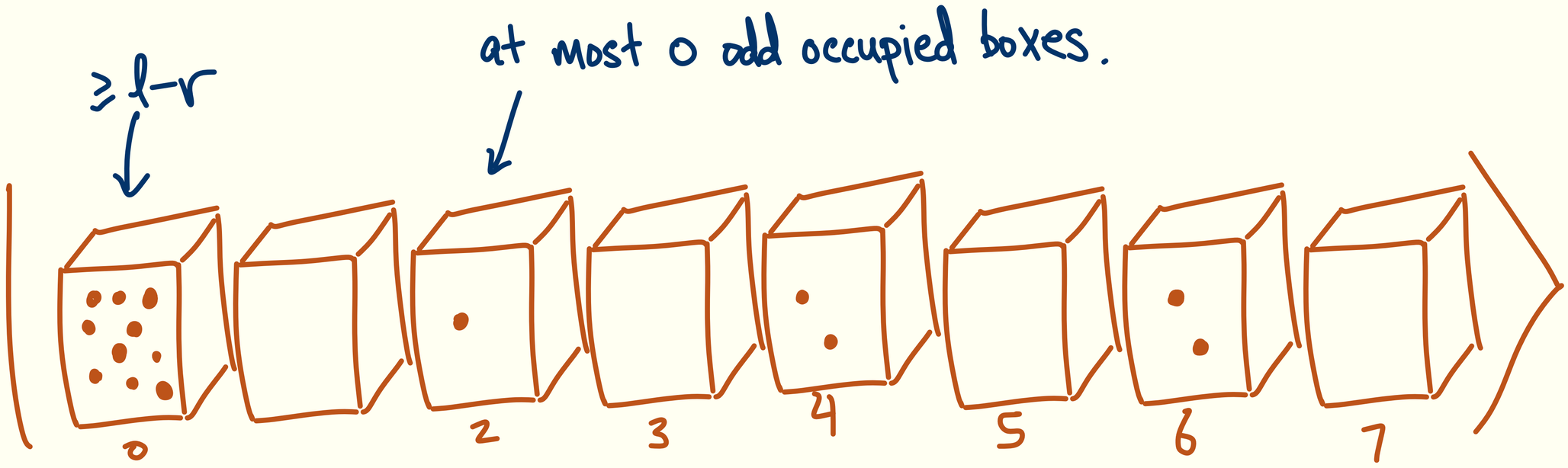
Quasi-even: At most o of the non-zero indices are odd.



Sampling bounds on quasi-even condensates

Claim: Let $|\psi\rangle$ be a state that is supported entirely on (r, o)-quasi-even condensate, then the following bound holds for all collections $z_1, \ldots, z_v \in \{0,1\}^n$:

$$\langle\psi| n_{z_1} \ldots, n_{z_\ell} |\psi\rangle \leq \left(\text{poly}(v, r) \cdot \frac{\sqrt{\ell}}{2^{n/4}} \right)^v$$

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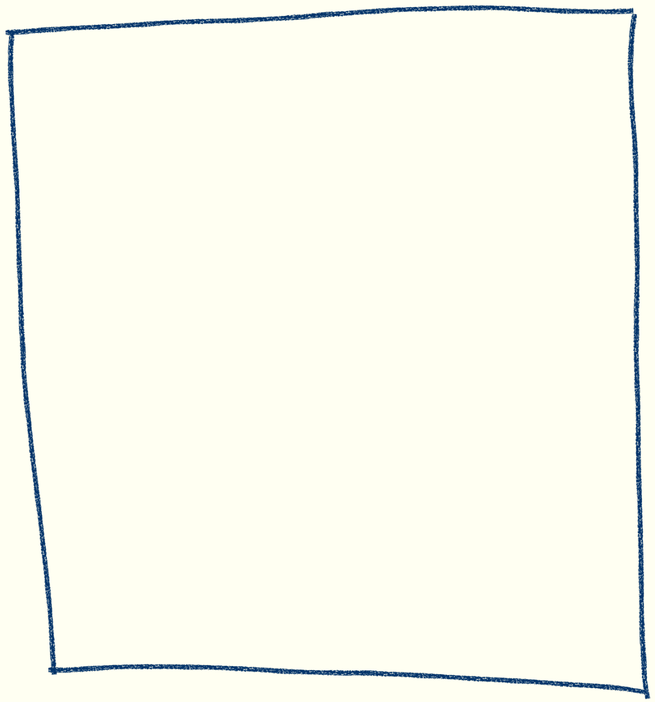
This number upper bounds the sampling success probability (applying Markov's inequality) → proving that the "purified" state of any algorithm is a quasi-even condensate would complete the proof.

The code intersection problem

Here is how we can sample an instance of the code intersection problem:



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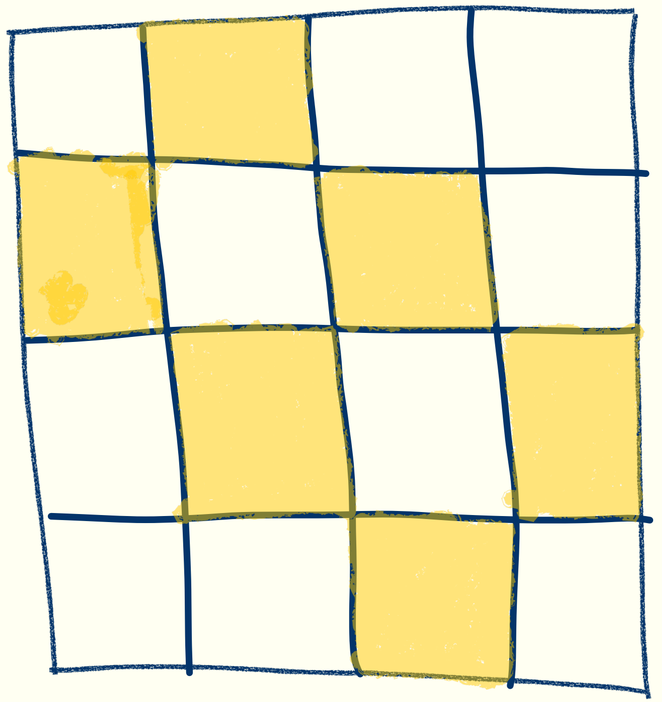

$(q=4, n=2)$

$s=1$

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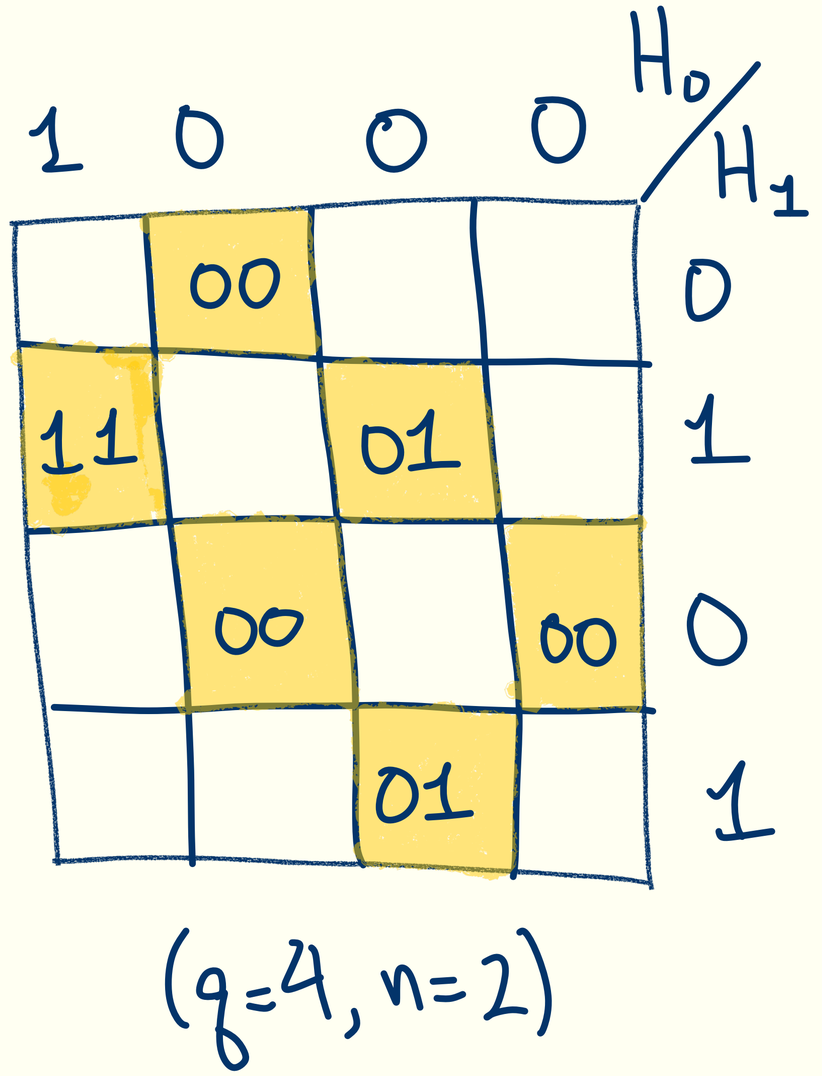

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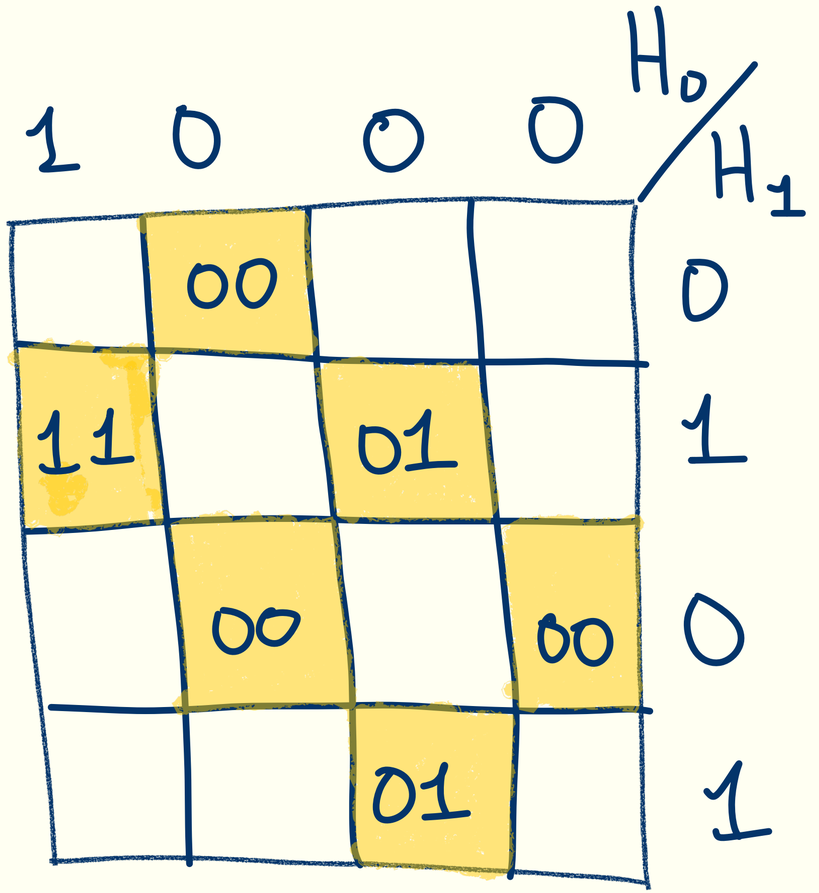
3. Let the set be $S_{C,H} = \{(x, c) : c \in C \text{ and } H_i(c_i) = x_i \ \forall i\}$



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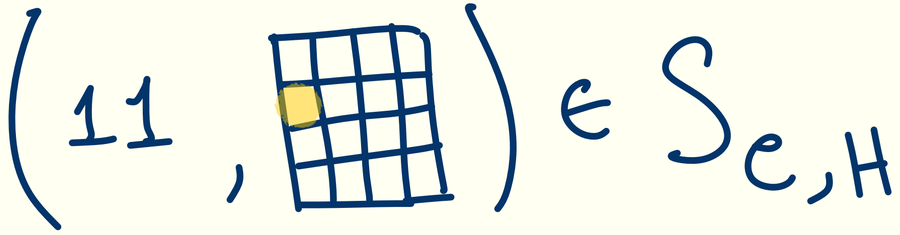

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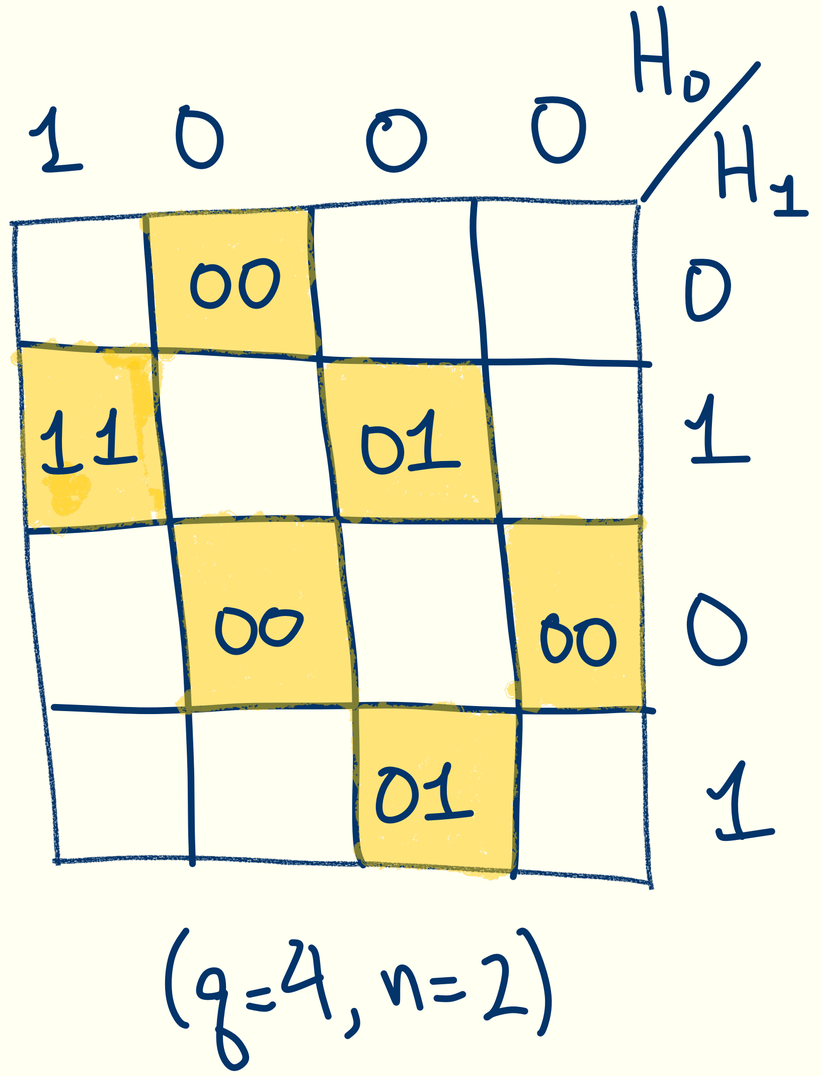
$S_{C,H} = \{(x, v) : v \in C \text{ and } H_i(v_i) = x_i \ \forall i\}$

Input: Oracle access to $S_{C,H} \cap E$ for $E \subseteq \{0,1\}^n \times (\mathbb{F}_q^s)^n$



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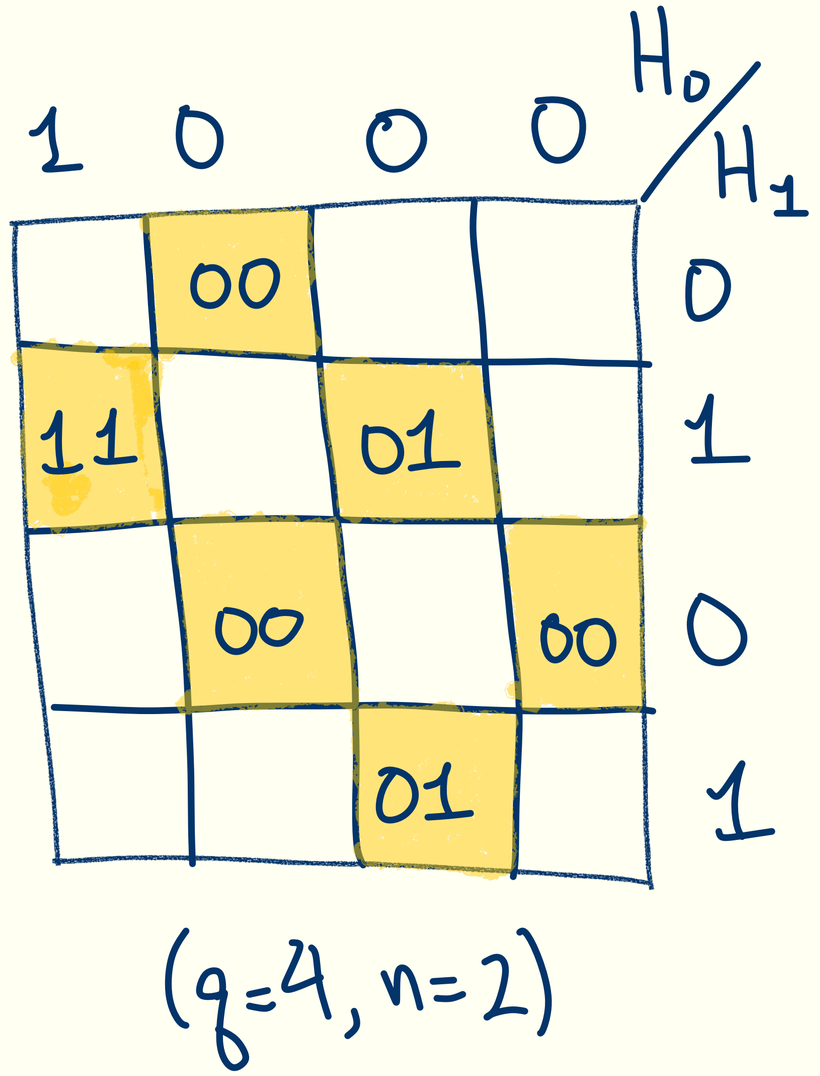

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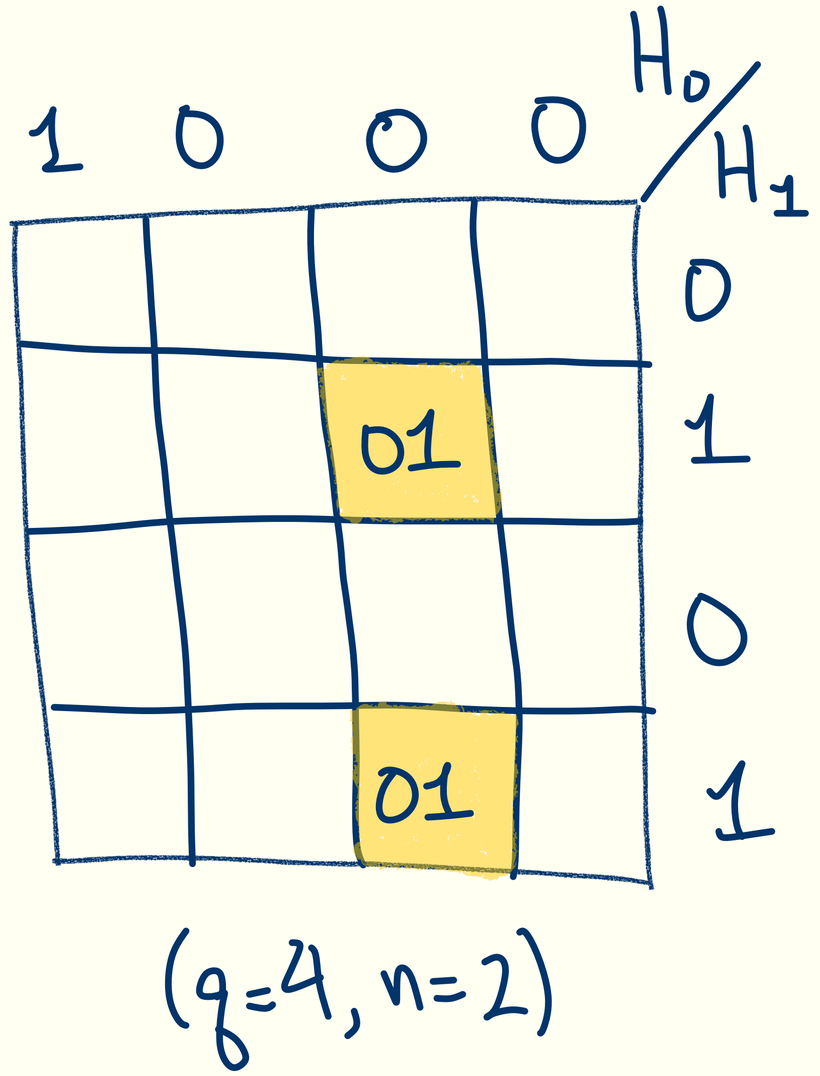
(No) at most $1/3$ of $x's$ are in $S_{C,H} \cap E$, i.e., $E \subseteq F \times (\mathbb{F}_q^s)^n$, for some $|F| \leq 2^n/3$, promised one is the case.



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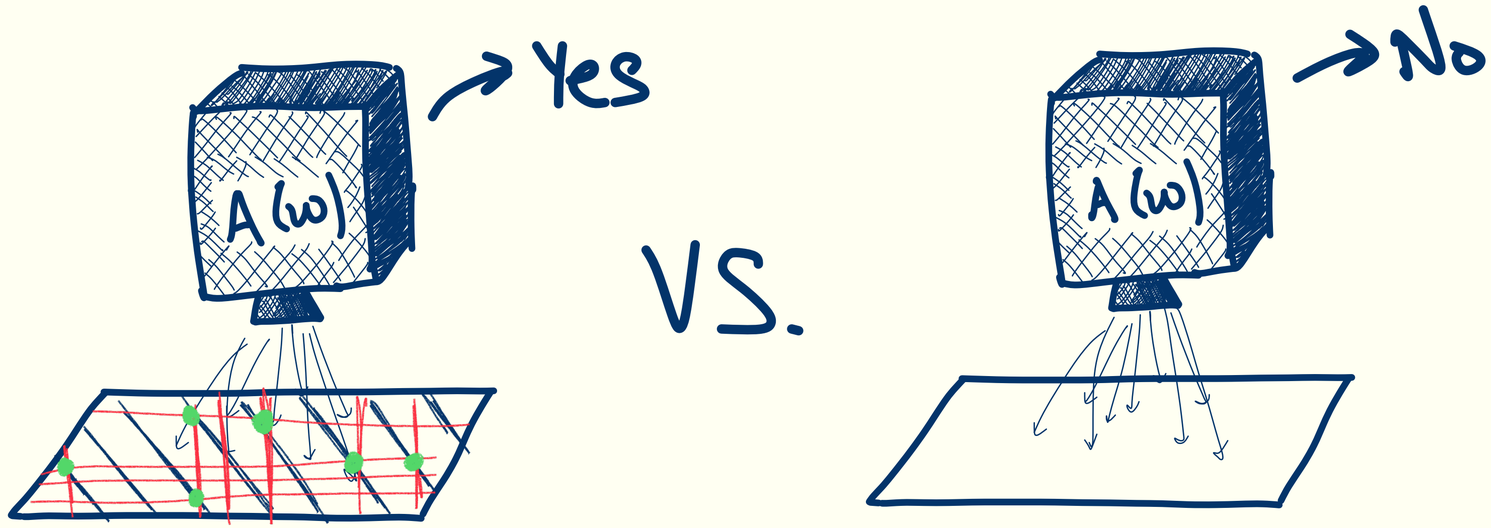
If the functions are random and the code has high enough distance, the Fourier transform of the witness corresponds to decodable errors.

Applying the sampler idea to code intersection

Theorem: If there is a QCMA algorithm, making t queries to $S_{C,H}$, and taking a witness of length w, then for all $0 \leq v \leq 2^n/3$, there is an algorithm making no queries that outputs v many pairs of distinct codewords and their hashes, (x, c) with probability

$$\geq 2^w \left(\frac{1}{36t^2}\right)^v$$



List recoverability and code intersection

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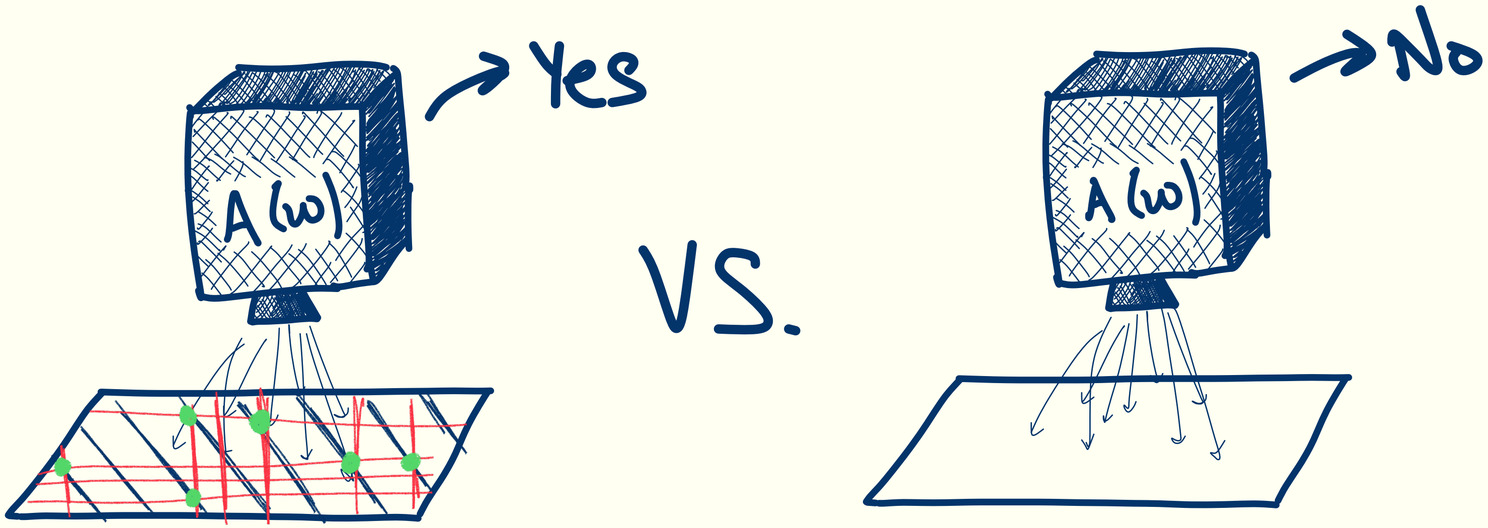
Definition: A code is (ℓ, L)-list recoverable if, given lists $S_1, \ldots, S_n$ of average size ℓ, the number of codewords you can build from them is at most L.

List recoverability and code intersection

If we instantiate our code intersection problem with a $(v/2,v)$-list recoverable code, then the sampler must guess at least $nv/2$ many hash values, so we can bound its success probability by

$$(1-p)^{nv/2} \approx \left(\frac{1}{2^{\lambda}}\right)^{v}$$



List recoverability and code intersection

Good news: There are families of codes (we take multiplicity codes) that have good list recovery up to $v = 2^n$ many codewords, so we can take those in our code.

$$f(X) \mapsto \begin{pmatrix} f(\alpha_1), \ldots, f(\alpha_n) \\ f'(\alpha_1), \ldots, f'(\alpha_n) \\ f^{(2)}(\alpha_1), \ldots, f^{(2)}(\alpha_n) \\ \cdots \end{pmatrix}$$

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Turns out this can be solved without too many modifications to the YZ algorithm.

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  → In the code intersection problem, just adding the derivatives of the polynomial gives us the increased list recovery needed to make the lower bound work.
- **Much more work is needed!**
  → Understanding oracles with structure seems to require an understanding that structure, seem to be annoying to deal with using general methods.
  → We still don't really understand all the "kinds" of structure that a quantum proof can help identify that a classical proof can't.

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  → Our ideas lie in the intersection of ideas used for quantum money (subset states ↔ subspace states, Fourier transform of S ↔ Fourier transform for group actions).
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- **Is there a connection to the Aaronson-Ambainis conjecture?**
  → Both Liu-Mutreja-Yuen'24 and Zhandry'24 showed that there is a connection between QCMA versus QMA and pseudorandomness against quantum algorithms.
  → Our proof didn't say anything about this, but could you use our techniques?

Thanks for listening!