PCMI topological aspects of quantum codes, problem session #2

Instructor: Jeongwan Haah, Teaching Assistant: John Bostanci

1. **(The Syndrome and its Generalization)** We define the syndrome to be a map taking a Pauli Z operator to a bit vector (one for each X-stabilizer generator), defined by the commutation relation. Show that this is a $\mathbb{Z}_2$-linear map.

   Define the syndrome for a code over prime p-dimensional qudits with $X = \sum_{j \in \mathbb{Z}_p} |j+1\rangle\langle j|$ and $Z = \sum_{j \in \mathbb{Z}_p} e^{2\pi i j/p} |j\rangle\langle j|$ and show that it is a $\mathbb{Z}_p$-linear map.

2. **(The $4D$ Toric Code.)** Imagine a $4D$ toric code on a hypercubic lattice where qubits are associated with 2-cells (i.e. faces).

   (a) How many Z checks act on every qubit?

   (b) What are the left and right degrees of the associated Tanner graph?

3. **(Visualizing the $3D$ Toric Code.)** Show (visually) that in the $3D$ toric code, two logical X operators in the same cohomology class multiply to form a stabilizer.

4. **(Entanglement Renormalization.)** Let $\mathcal{S}$ be a Pauli stabilizer group generated by $Q_1, \ldots, Q_s$ and P be a Pauli operator that commutes with Q_j for all $j \neq 1$. In class we showed that the applying the unitary $\frac{1}{\sqrt{2}}(P + Q_1)$ yields the post-measurement stabilizer when measuring P and seeing the outcome $+1$.

   (a) What happens when we apply the unitary $\frac{1}{\sqrt{2}}(P - Q_1)$?

   (b) Is this a Clifford unitary?

5. **(Commuting Circuits on Lattices.)** Consider a quantum circuit (of any depth) acting on qubits arranged in a D-dimensional Euclidean lattice. Say that all of the gates must commute with each other[1], and every gate is constrained to act on a set of qubits that are within some ball of constant radius. Show that this circuit can be implemented by a depth $O(D+1)$ quantum circuit.

References

[BJS10] Michael J. Bremner, Richard Jozsa, and Dan J. Shepherd. Classical simulation of commuting quantum computations implies collapse of the polynomial hierarchy. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 467(2126):459–472, aug 2010.

[1] Although the requirement that all gates commute with each other might seem like it only yields “simple” circuits, this model of computation, called IQP, has interesting properties. For example, it can not be sampled from efficiently unless PH collapses [BJS10], and the GHZ state can be prepared in IQP (and it is known that it can not be prepared in QNC_0 [WKST19])

[WKST19] Adam Bene Watts, Robin Kothari, Luke Schaeffer, and Avishay Tal. Exponential separation between shallow quantum circuits and unbounded fan-in shallow classical circuits. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2019, page 515–526, New York, NY, USA, 2019. Association for Computing Machinery.