# PCMI topological aspects of quantum codes, problem session #2

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1. **(The Syndrome and its Generalization)** We define the syndrome to be a map taking a Pauli $Z$ operator to a bit vector (one for each $X$-stabilizer generator), defined by the commutation relation. Show that this is a $\mathbb{Z}_2$-linear map.

   Define the syndrome for a code over prime $p$-dimensional qudits with $X = \sum_{j \in \mathbb{Z}_p} |j+1\rangle\langle j|$ and $Z = \sum_{j \in \mathbb{Z}_p} e^{2\pi i j/p} |j\rangle\langle j|$ and show that it is a $\mathbb{Z}_p$-linear map.

   **Solution:** We can easily check that the commutation relation satisfies the relationship that we want, let $A$ and $B$ be two paulis, and $S$ be a stabilizer element, then

   $$ABS = (-1)^{[B,S]} ASB = (-1)^{[B,S]+[A,S] \bmod 2} SAB \,.$$

   Here I am using the notation that $[A, B]$ outputs $0$ if the two commute, and $1$ if they anti-commute (how the syndrome would). Thus, the syndrome adds as we expect $(\mathrm{mod} 2)$.

   To define a syndrome for prime $p$-dimensional qudits, we can define the commutator (in our abused notation here) to output a number from $0$ through $p-1$, corresponding to:

   $$[A, B] = k \iff AB = e^{2\pi i k/p} BA \,.$$

   The same equation (above) shows that the commutator is $\mathbb{Z}_p$-linear, which is what we want.

2. **(The $4D$ Toric Code.)** Imagine a $4D$ toric code on a hypercubic lattice where qubits are associated with 2-cells (i.e. faces).

   (a) How many $Z$ checks act on every qubit?

   (b) What are the left and right degrees of the associated Tanner graph?

   **Solution:** Intuition about $4D$: say that every $0$-cell (point) in the $4D$ toric code is given by set of $4$ integers, $(a, b, c, d)$. Then we can see that we can count the edges coming out of a point by fixing $3$ of the dimensions, and taking the $4$'th to be the axis the edge is a long, and then we get $2$ directions for the edge to go on (plus $1$ or minus $1$), for a total of $8$ edges coming out of a point. Similarly, we count faces by picking $2$ free directions, and taking all $4$ choices of $\pm 1$ to add, and furthermore, for a fixed face, we get all of the volumes coming out of it by further fixing a third dimension (2-choices), and picking either plus or minus $1$ (2-choices).

   (a) Thus, there are $4$ $Z$-checks (volumes) acting on every qubit (faces). Note that there are also $4$ $X$-checks (edges) acting on every qubit (face), which can be seen just by looking at a face and seeing it has $4$ edges on it.

(b) The left and right degrees correspond to how many qubits every stabilizer touches. Here we see that every $Z$-check (volume) touches $6$ qubits (faces) by inspection, and using the same logic, every $X$ check touches $6$ qubits (faces), which comes from taking an edge, choosing a second free direction ($3$ choices), and choosing either plus or minus $1$ ($2$ choices).

3. **(Visualizing the $3D$ Toric Code.)** Show (visually) that in the $3D$ toric code, two logical $X$ operators in the same cohomology class multiply to form a stabilizer.

4. **(Entanglement Renormalization.)** Let $\mathcal{S}$ be a Pauli stabilizer group generated by $Q_1, \ldots, Q_s$ and $P$ be a Pauli operator that commutes with $Q_j$ for all $j \neq 1$. In class we showed that the applying the unitary $\frac{1}{\sqrt{2}}(P + Q_1)$ yields the post-measurement stabilizer when measuring $P$ and seeing the outcome $+1$.

   (a) What happens when we apply the unitary $\frac{1}{\sqrt{2}}(P - Q_1)$?

   (b) Is this a Clifford unitary?

**Solution:** To see the action of this unitary of the stabilizer, first not that for all $Q_j$ for $j \neq 1$, the unitary commutes with $Q_j$, so measuring it does not change the stabilizer. We can compute what happens to the Pauli operator $Q_1$ after being conjugated by the unitary:

$$\begin{aligned}\frac{1}{2}(P - Q_1)Q_1(P - Q_1) &= \frac{1}{2}(PQ_1P - Q_1Q_1P - PQ_1Q_1 + Q_1Q_1Q_1) \\ &= \frac{1}{2}(-Q_1 - P - P + Q_1) \\ &= -P\,.\end{aligned}$$

Here we use the commutation relation between Pauli operators, and the fact that $Q_1^2 = P^2 = \mathrm{id}$. Thus, doing that unitary maps $Q_1$ to $-P$, corresponding to measuring $P$ and seeing the outcome $-1$.

This is a Clifford unitary, to see why, take any Pauli $P$, and consider conjugating by it. Note that since $P$ and $Q_1$ anti-commute, $A$ must anti-commute with exactly $1$ of $P$ and $Q_1$.

$$\begin{aligned}\frac{1}{2}(P - Q_1)A(P - Q_1) &= \frac{1}{2}(PAP - QAP - PAQ + QAQ) \\ &= \frac{1}{2}([P, A]A - [Q, A]AQP + [P, A]AQP + [Q, A]A) \quad = -AQP\,.\end{aligned}$$

Here we just wrote out the expression and used the fact that $A$ anti-commutes with exactly $1$ of $P$ and $Q_1$, so the commutators have opposite signs (here taking the commutator to be $\pm 1$ depending on if they commute, not the usual group commutator). Thus, $A$ is mapped to another Pauli, so the unitary is Clifford.

5. **(Commuting Circuits on Lattices.)** Consider a quantum circuit (of any depth) acting on qubits arranged in a $D$-dimensional Euclidean lattice. Say that all of the

gates must commute with each other[1], and every gate is constrained to act on a set of qubits that are within some ball of constant radius. Show that this circuit can be implemented by a depth $O(D+1)$ quantum circuit.

**Solution:** At a high level, here is the strategy for constructing the $D+1$ depth quantum circuit. The first step requires the existance of tilings of $D$-dimensional Euclidean space that are both $D+1$ colorable, and for which regions of the same color are separated, for example:

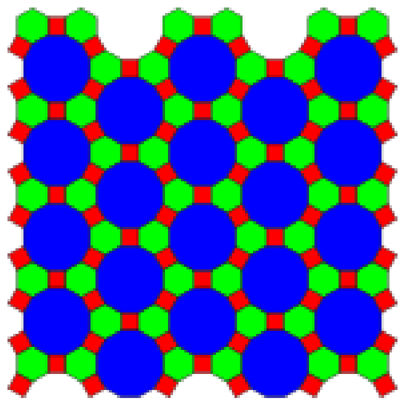

Figure 1: A 3-colorable tiling of $2D$-space where every color is separated.

(a) Pick a tiling of $D$-dimensional Euclidean space that is $D+1$-colorable, and every colored region is separated by at least twice the constant radius, $r$. We will also assume that every one of the closed balls is a constant size (in the size of the grid).

(b) Now pick a color, and look at all of the qubits in that region. Since all of the gates act in a ball of constant radius, every qubit in that region must only interact with qubits in the expanded region, so we can construct a circuit that outputs the state of the first region (not the expanded one), that only acts on qubits in the expanded region, which is constant size. Thus, in constant depth we can implement a circuit that maps the input to the output on the region of this color.

(c) Now repeat this for every color, and we get a depth $O(D+1)$ circuit. Since everything commutes it does not matter what order we choose to do this in.

And if you allow yourself to (say) implement any constant qubit unitary in a single gate, then you get an exactly depth $D+1$ circuit.

[1] Although the requirement that all gates commute with each other might seem like it only yields "simple" circuits, this model of computation, called IQP, has interesting properties. For example, it can not be sampled from efficiently unless PH collapses [BJS10], and the GHZ state can be prepared in IQP (and it is known that it can not be prepared in $\mathsf{QNC}_0$ [WKST19])

# References


[BJS10] Michael J. Bremner, Richard Jozsa, and Dan J. Shepherd. Classical simulation of commuting quantum computations implies collapse of the polynomial hierarchy. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 467(2126):459–472, aug 2010.

[WKST19] Adam Bene Watts, Robin Kothari, Luke Schaeffer, and Avishay Tal. Exponential separation between shallow quantum circuits and unbounded fan-in shallow classical circuits. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2019, page 515–526, New York, NY, USA, 2019. Association for Computing Machinery.