PCMI topological aspects of quantum codes, problem session #4

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1. **(Correcting the T Injection.)** In the T injection circuit, we prepare an ancilla in the $|T\rangle$ state, and then measure ZZ on the ancilla and target, and finally measure X on the ancilla qubit. Write the post-measurement state of the system after all 4 possible measurement outcomes (00, 01, 10, 11), and determine the corrections needed, ignoring global phase factors.

   Solution: Assume that we have a qubit in the state $\alpha|0\rangle + \beta|1\rangle$, then we can write the overall state of the system as:

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\pi/4}|1\rangle\right) \otimes (\alpha|0\rangle + \beta|1\rangle)$$
$$= \frac{1}{\sqrt{2}}\left(\alpha|00\rangle + \alpha e^{-i\pi/4}|01\rangle + \beta|01\rangle + \beta e^{-i\pi/4}|11\rangle\right) .$$

   First let us assume that we measure ZZ and see the $+1$ eigenvalue, then we will have projected onto $|00\rangle$ and $|11\rangle$, so we will have the state

$$\alpha|00\rangle + \beta e^{-i\pi/4}|11\rangle = \alpha|0\rangle \otimes \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) + \beta e^{-i\pi/4}|1\rangle \otimes \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) .$$

   Thus, when we measure X on the second qubit and see $+1$, we are going to get the state

$$\alpha|0\rangle + \beta e^{-i\pi/4}|1\rangle ,$$

   and no correction is needed. When we see -1, we get the state

$$\alpha|0\rangle - \beta e^{-i\pi/4}|1\rangle ,$$

   and applying a Z will get us to the desired state. Now let us consider the ase when we measure ZZ in the start and see the -1 eigenvalue, which corresponds to projecting onto $|01\rangle$ and $|10\rangle$. The state of the system will be

$$\beta|01\rangle + \alpha e^{-i\pi/4}|10\rangle = \beta|0\rangle \otimes \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\,\alpha e^{-i\pi/4}|1\rangle \otimes \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) .$$

   When we measure X and see a $+1$, we will have the following state

$$\beta|0\rangle + \alpha e^{-i\pi/4}|1\rangle .$$

   We can see that the correction needed is to first apply a rotation by $\pi/2$ (double the angle of the T gate), which is a S gate, and then apply an X gate to flip 0 to 1. Thus our correction is XS. If we measure a -1 eigenvalue, our state is

$$-\beta|0\rangle + \alpha e^{-i\pi/4}|1\rangle .$$

   Similarly, we can correct this by first applying S, then applying X and Z to fix the bit and phase flip errors. The point is that all of the corrections needed are Clifford gates.

2. **(Faulty T Injection.)** What happens if instead of $|T\rangle$, the T injection circuit is run with the state $Z|T\rangle$? Consider perfect Clifford operations but nonideal T ancilla state. What is the noise on the data qubit after the injection? What Clifford twirling was applied to the ancilla?

   Solution: Recall that $|T\rangle = T|+\rangle$. Since Z and T commute, we will end up with the state $T|-\rangle$ instead, which will basically flip the result of the last X check.

3. **(Level 3 Divisible Implies Triorthogonal.)** Recall the definitions of level-3 divisibility and triorthogonal:

   (a) A vector subspace V of m bits, equipped with an odd integer vector, t, is *level-*$(3,t)$ *divisible* if for all $v \in V$, $v \cdot t \mod 8 = 0$.

   (b) A $(k+s)\times n$ binary matrix is *triorthogonal* if the first (1) k rows have odd weight, (2) the last s rows have even weight, (3) the bitwise AND of every pair of rows has even weight, and (4) the bitwise AND of every triple of rows has even weight.

   Show that a level-3 divisible subspace has a triorthogonal generator matrix, where every row has even weight.

   Solution: Recall that $t \in \mathbb{Z}_8^m$ has all entries. Let $x, y \in V$, so $t \cdot x \equiv t \cdot y \equiv 0 \bmod 8$. Now, consider the following

   $$t \cdot (x+y) = t \cdot x + t \cdot y - 2t \cdot (x \wedge y) \equiv 0 \bmod 8\,.$$

   Subtracting $t \cdot x$ and $t \cdot y$, which are both $0 \bmod 8$, we see that $2t \cdot (x \wedge y) \equiv 0 \bmod 8$. Thus, $t \cdot (x \wedge y) \equiv 0 \bmod 4$. Similarly, we can repeat this this a third vector $z \in V$, so taking $t \cdot (x \wedge y) \equiv t \cdot z \equiv 0 \mod 4$ to find that the bitwise and of all three has even inner product with t.

   Since t has all odd entries, $t \cdot (x \wedge y) \bmod 2 \equiv \sum_i t_i \cdot (x \wedge y)_i \bmod 2 \equiv \sum_i (x \wedge y)_i \bmod 2 \equiv 0$, so we must have that $x \wedge y$ is even weight, and similarly for $x \wedge y \wedge z$. Thus, taking any orthogonal set of generators for V, we get a triorthogonal generator matrix where every row is even weight.

4. **(Level-4 Divisibility.)** Define level-4 divisibility, and show that the $3D$ color code's X stabilizer group is *not* level-4 divisible.