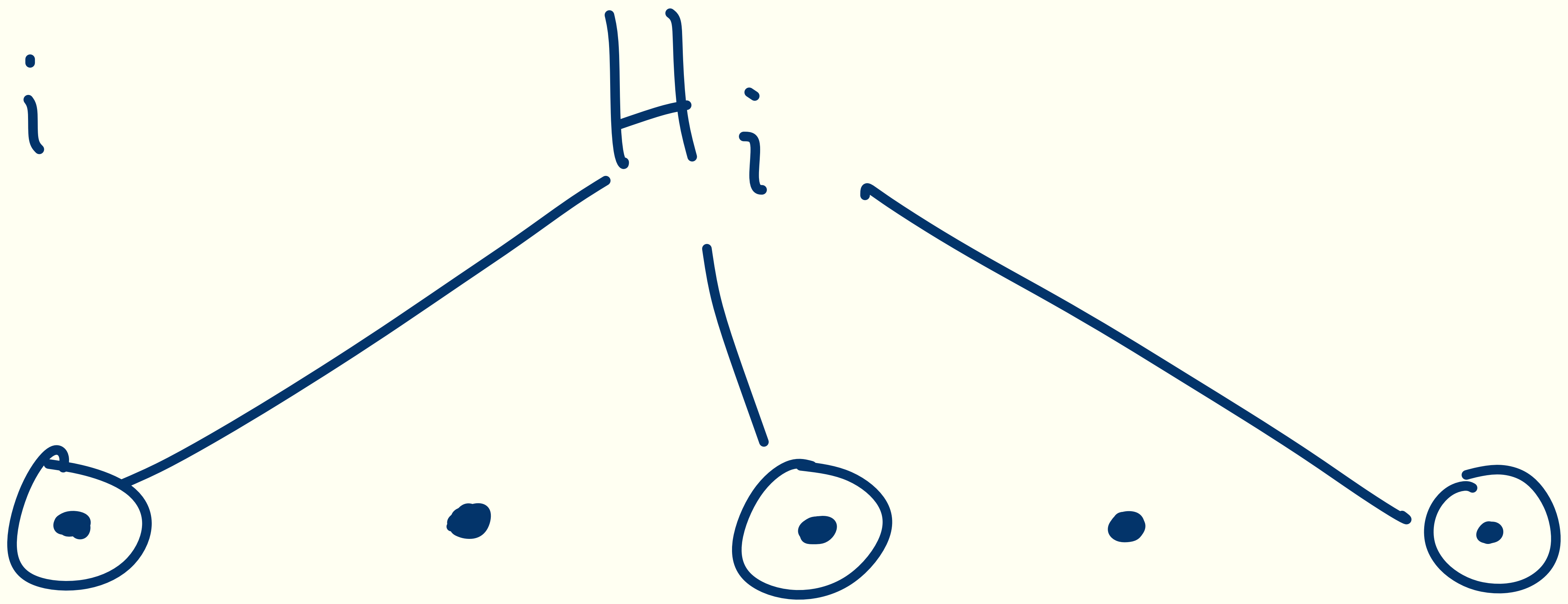
Commuting local Hamiltonians beyond 2D

John Bostanci, Yeongwoo Hwang

The local Hamiltonian problem

The local Hamiltonian problem is complete for QMA (quantum analog of NP).

$$H = \sum_i H_i$$



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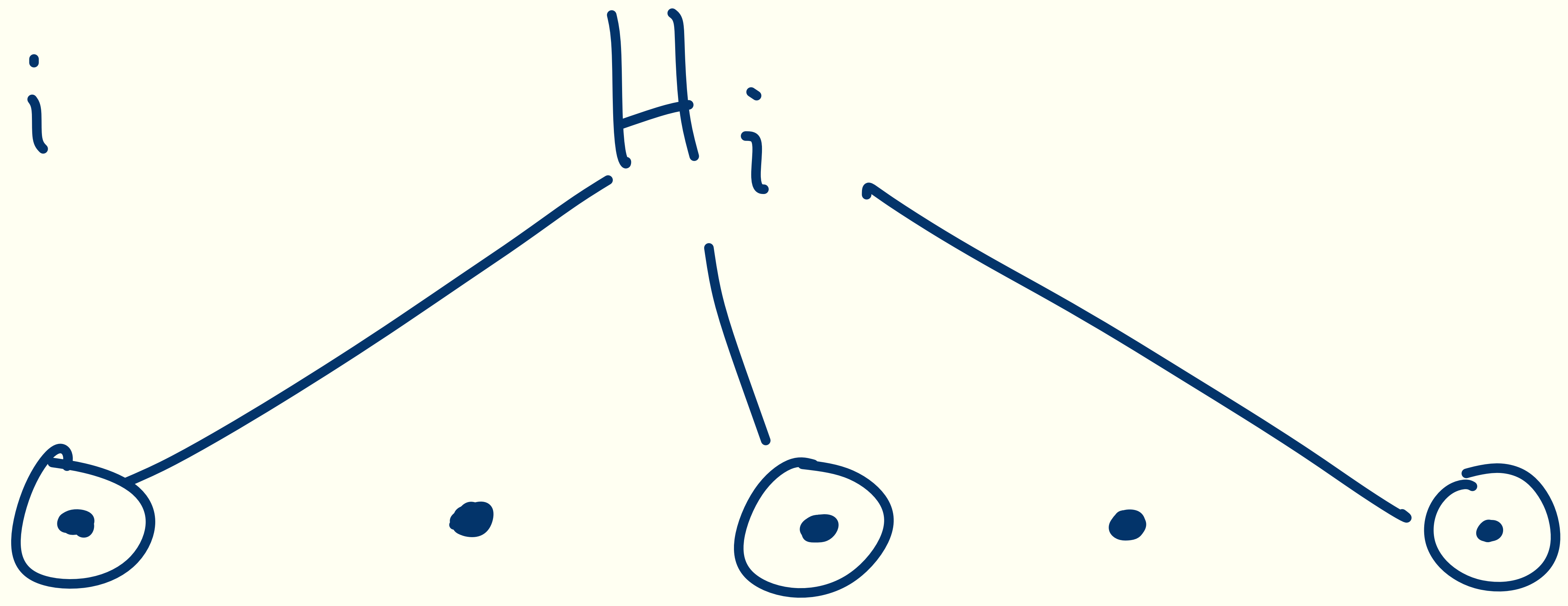

Decide if $\lambda_{min}(H) \leq \alpha$ or $\geq \beta$.

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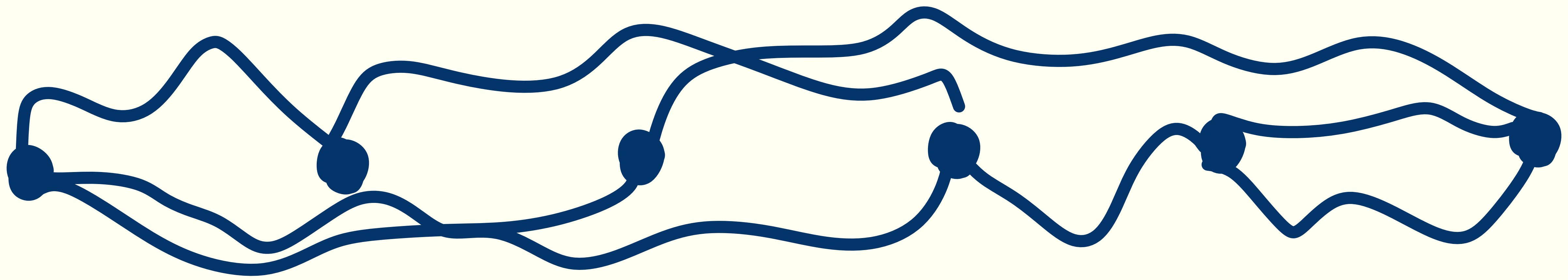

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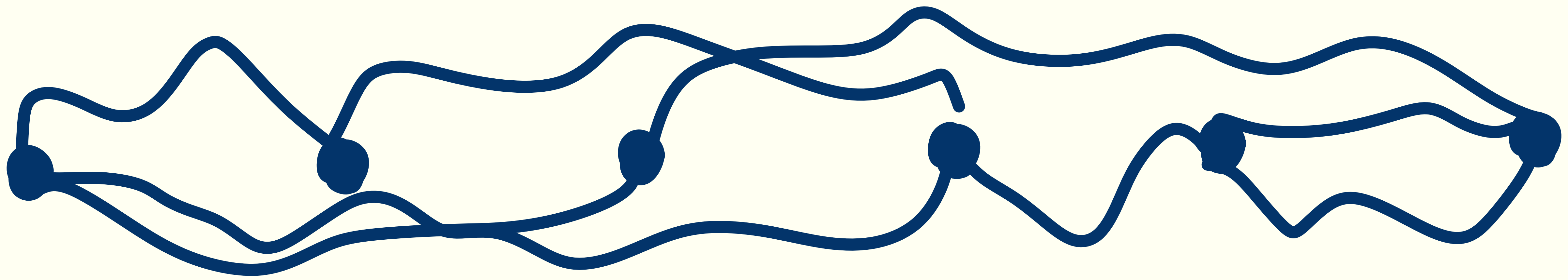
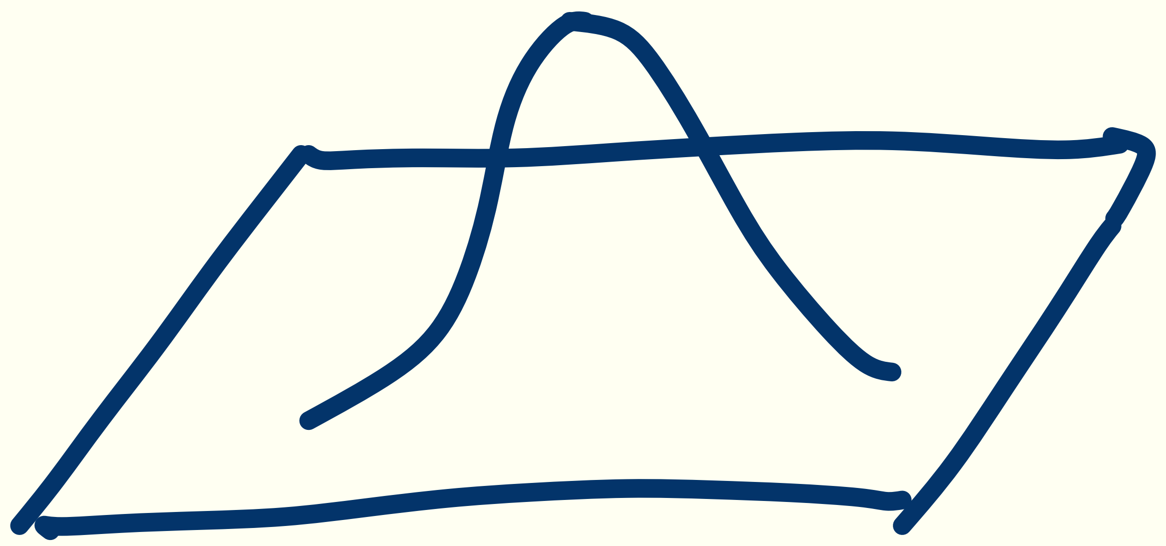
$|\psi\rangle \approx$ 

2. Non-commuting measurements give us the uncertainty principle.

$\Delta x \Delta p > 0$



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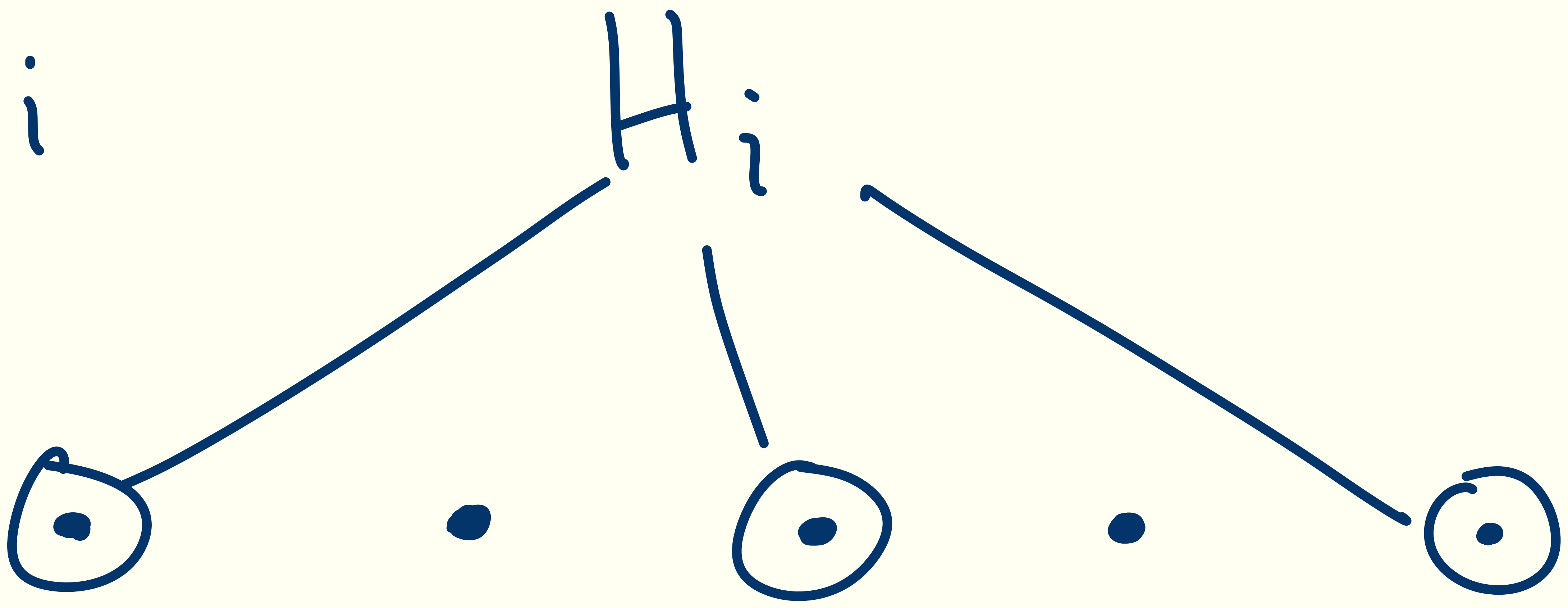

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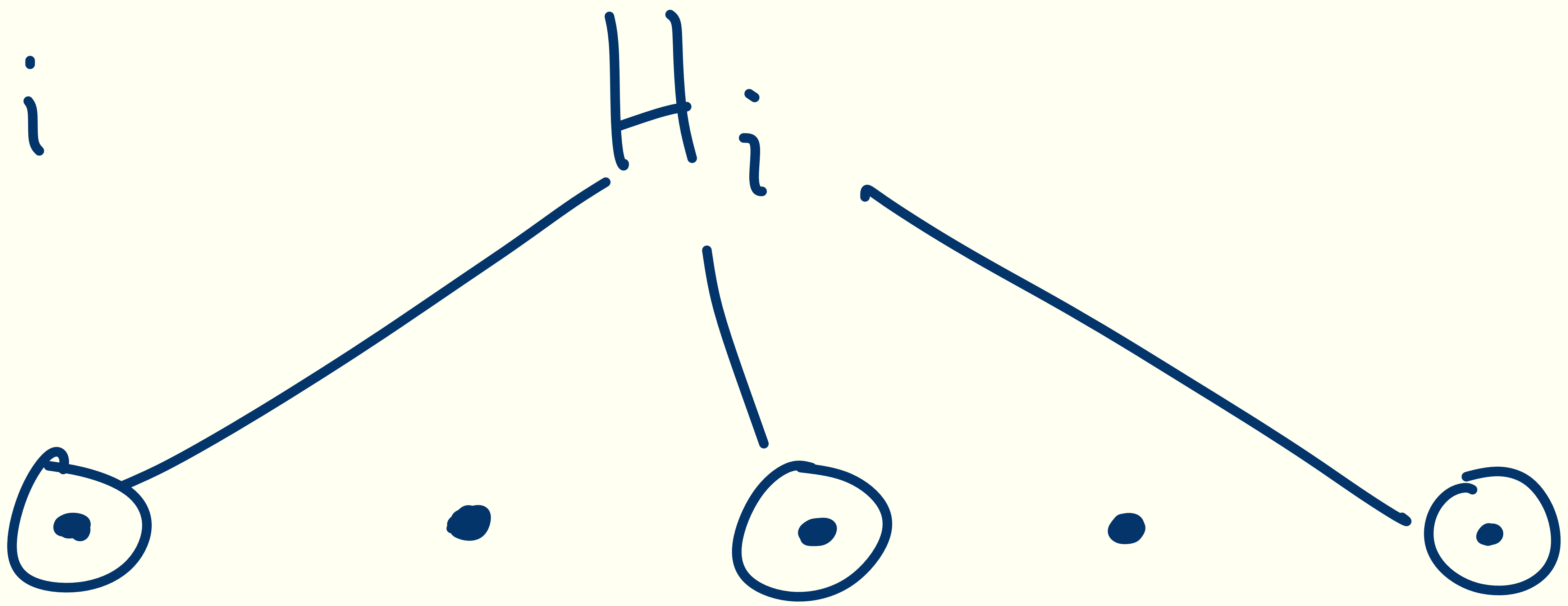

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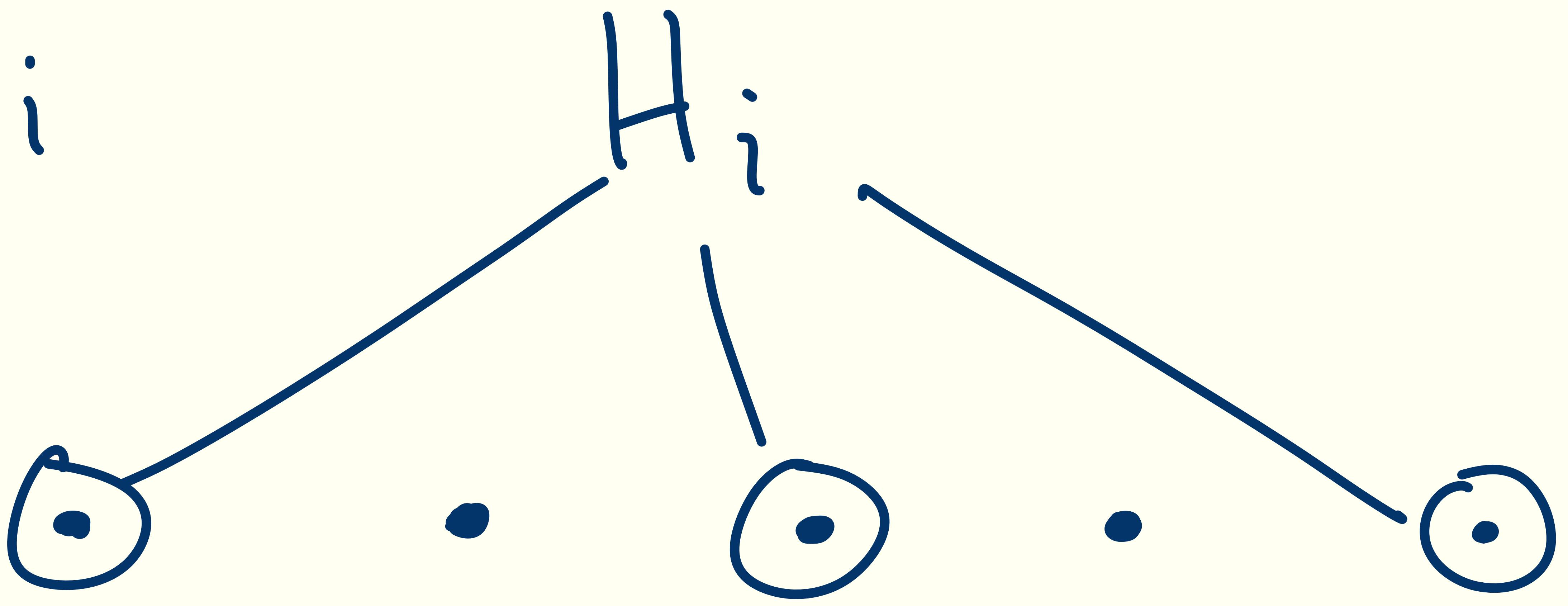

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What is the complexity of the CLH problem?

Appear in many places!

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- NLTS/Quantum PCP
- Thermalization

History of the commuting local Hamiltonians

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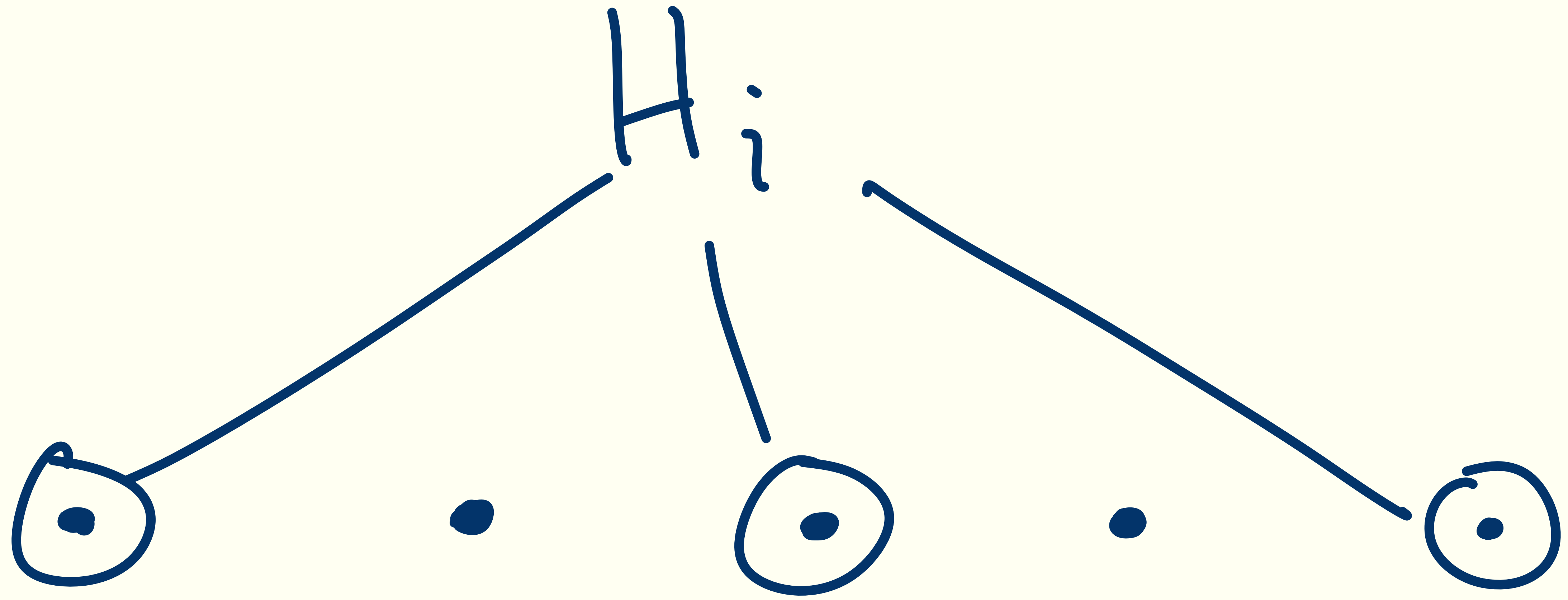
Rank-1 commuting local Hamiltonians

Every $H_i = |\psi_i\rangle\langle\psi_i|$



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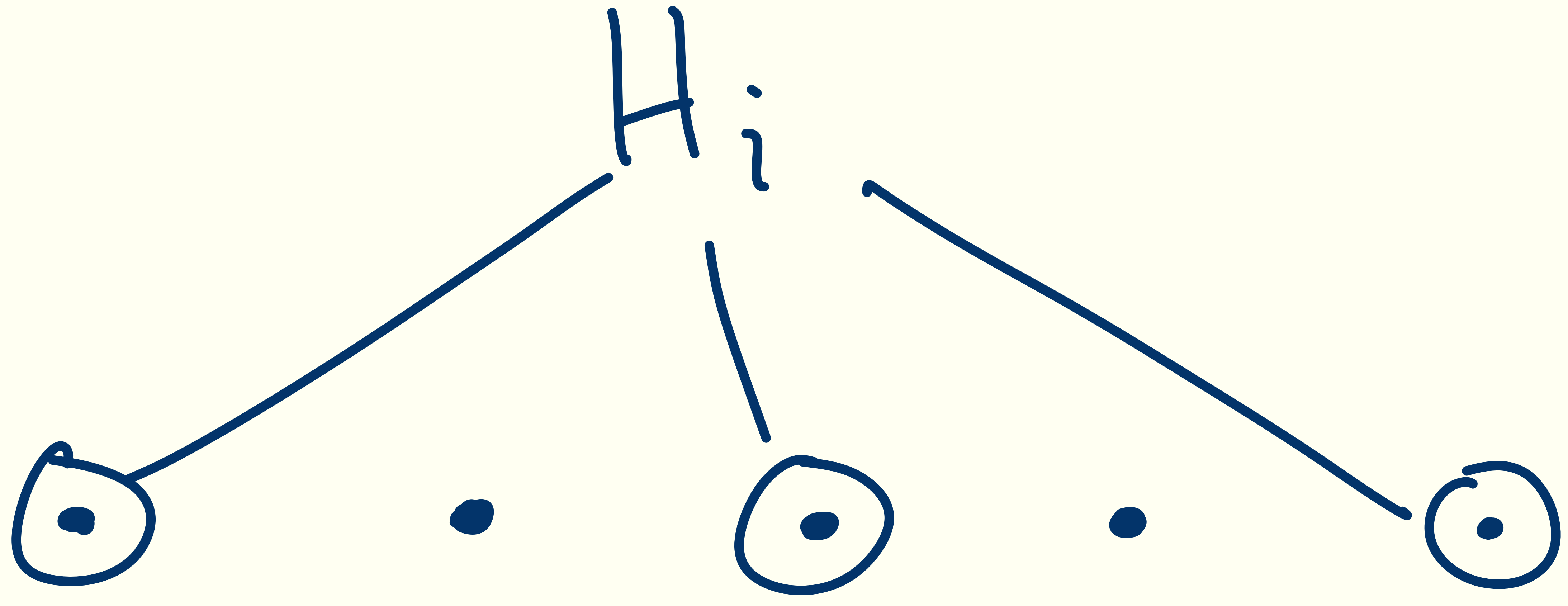

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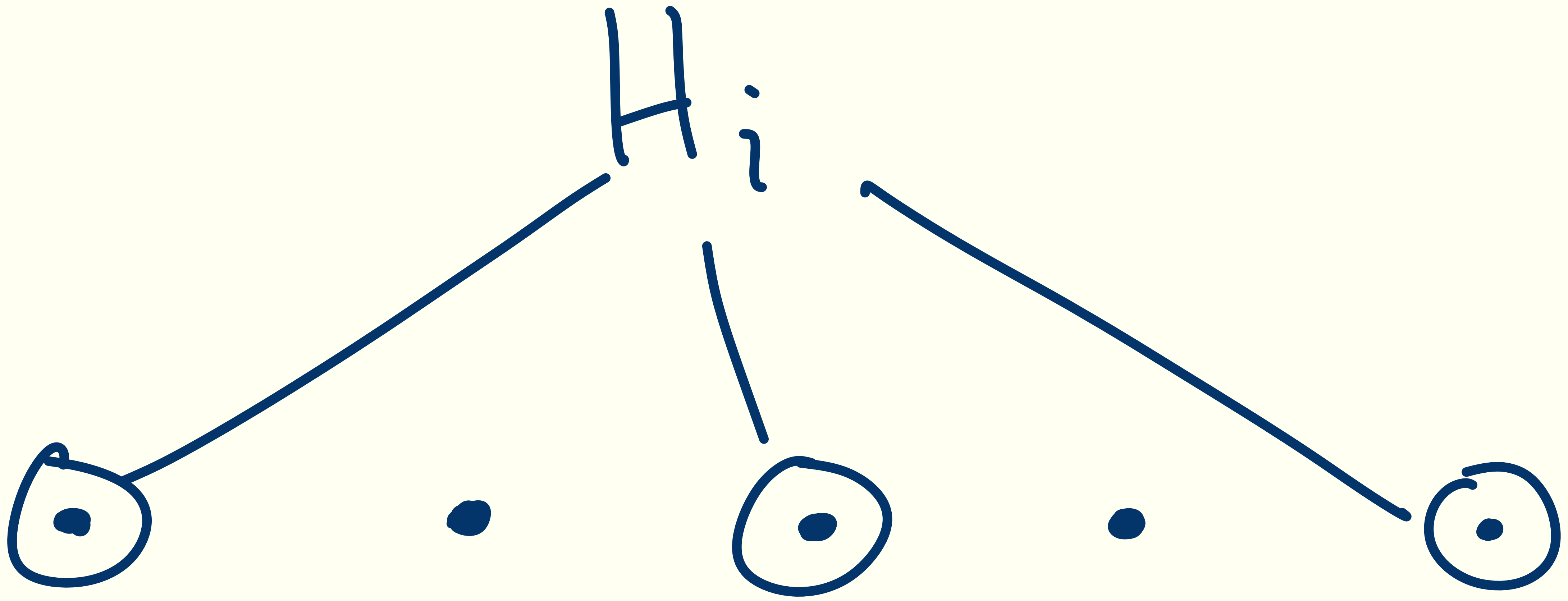
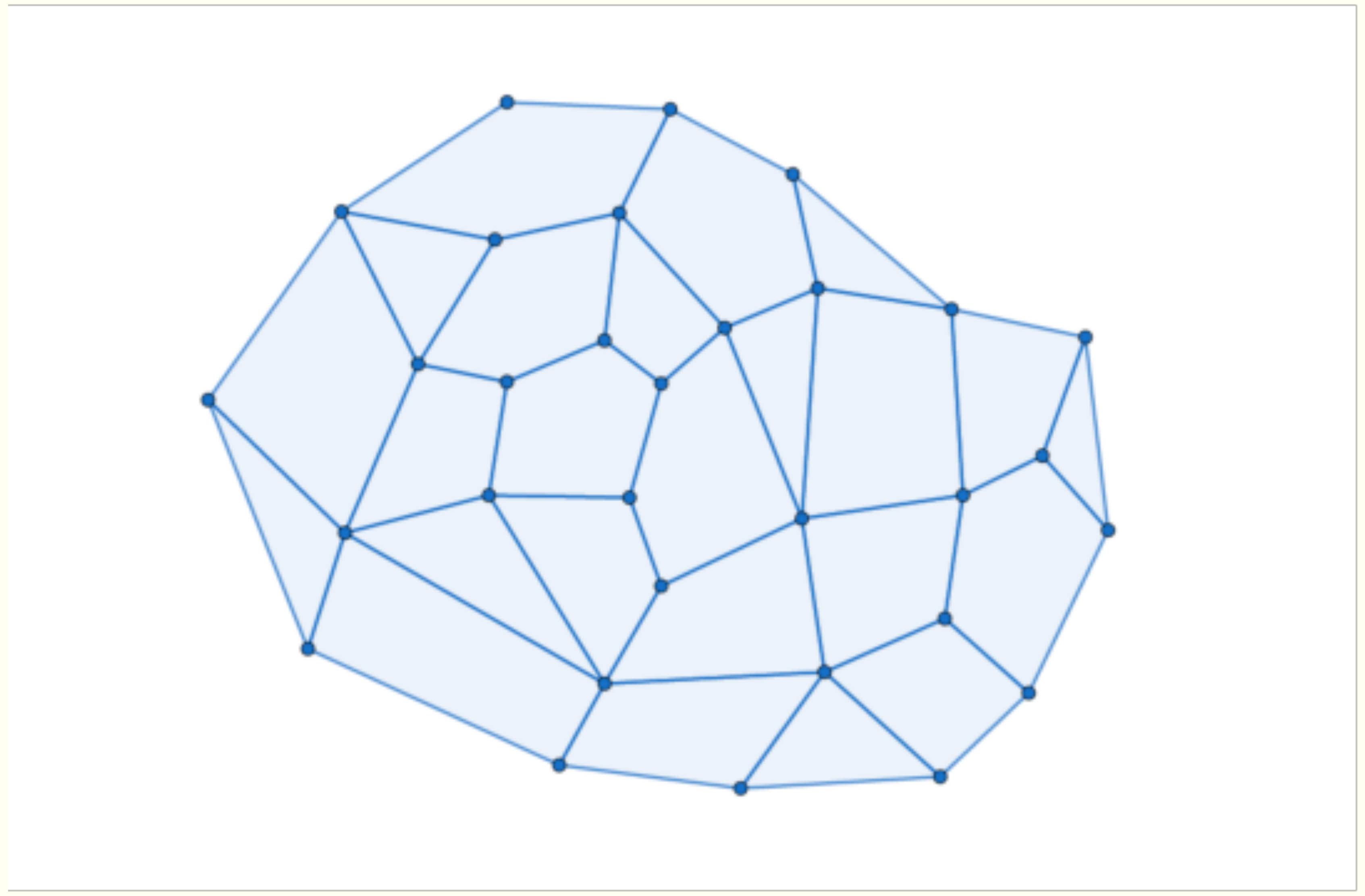


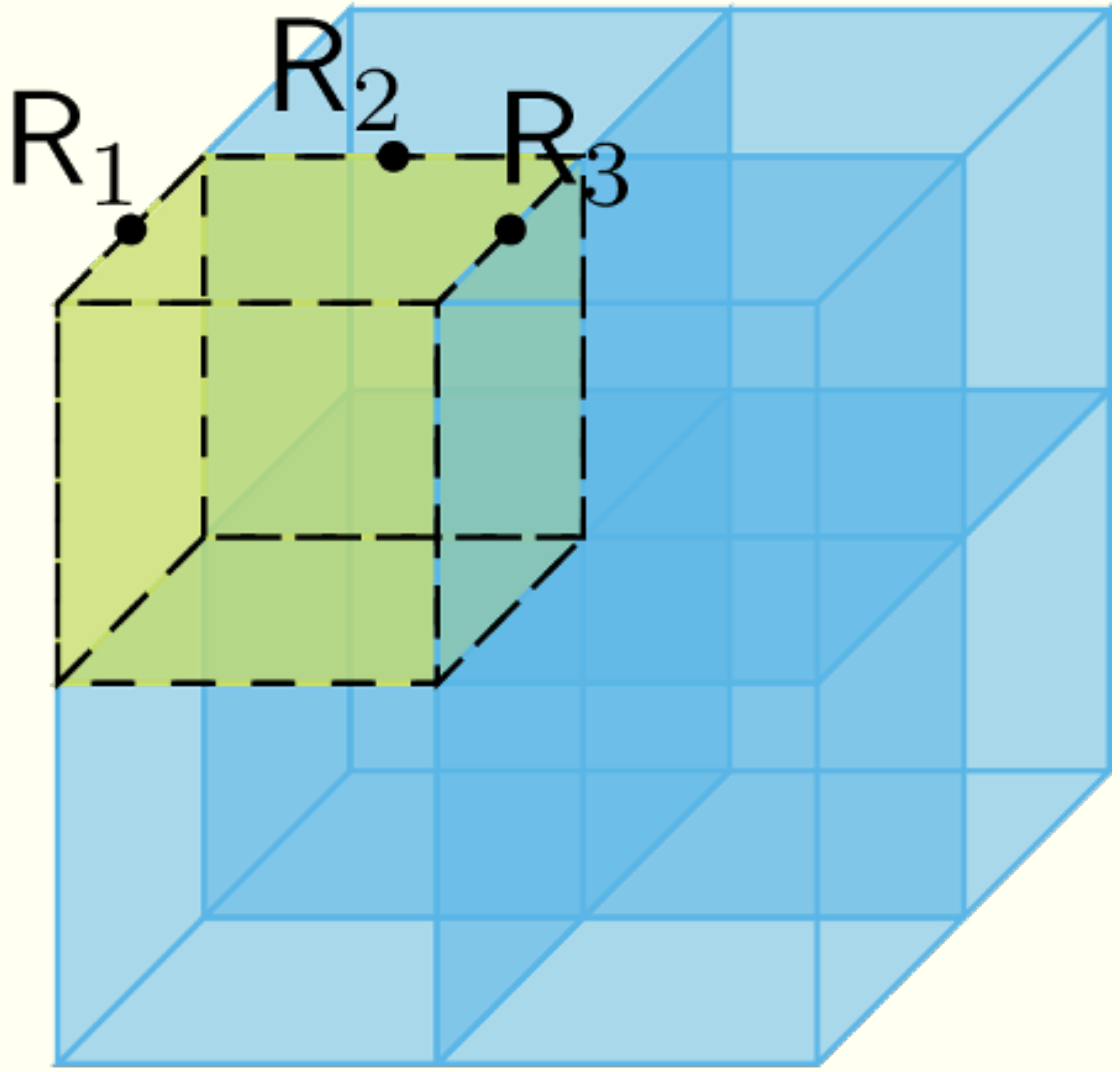
SAT: $x_1 \vee x_2 \vee x_3 = |000\rangle\langle 000|$

Heisenberg LH: $|EPR\rangle\langle EPR|$

Our results

- 2D-geometrically local, any qudit dimension, rank 1 CLH is in NP.
- 3D-geometrically local, on a cubic lattice, any qudit dimension, rank 1 CLH is in NP.





Background: The structure lemma [BV04]

The structure lemma tells us about how two Hamiltonians are “allowed” to commute!

Given two Hamiltonian terms, H_1 and H_2 that commute, they must look like:

$$P_1 = \begin{pmatrix} A_1 \otimes id & & \\ & A_2 \otimes id & \\ & & A_3 \otimes id \end{pmatrix} \qquad P = \begin{pmatrix} id \otimes B_1 & & \\ & id \otimes B_2 & \\ & & id \otimes B_3 \end{pmatrix}$$

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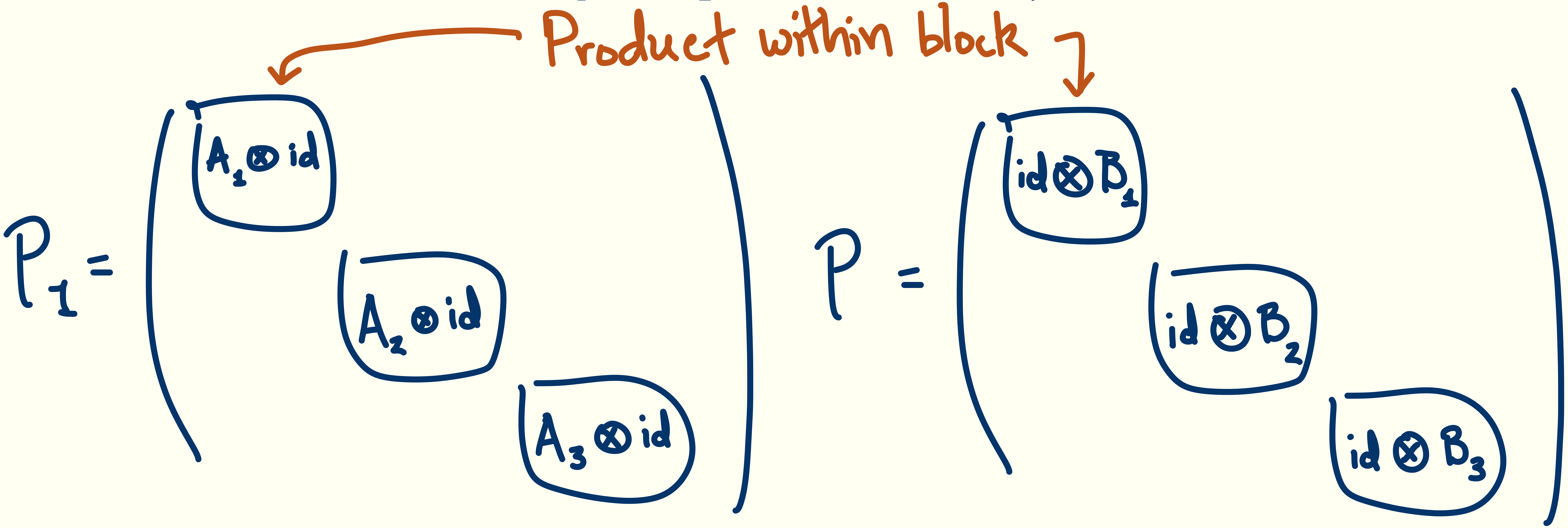
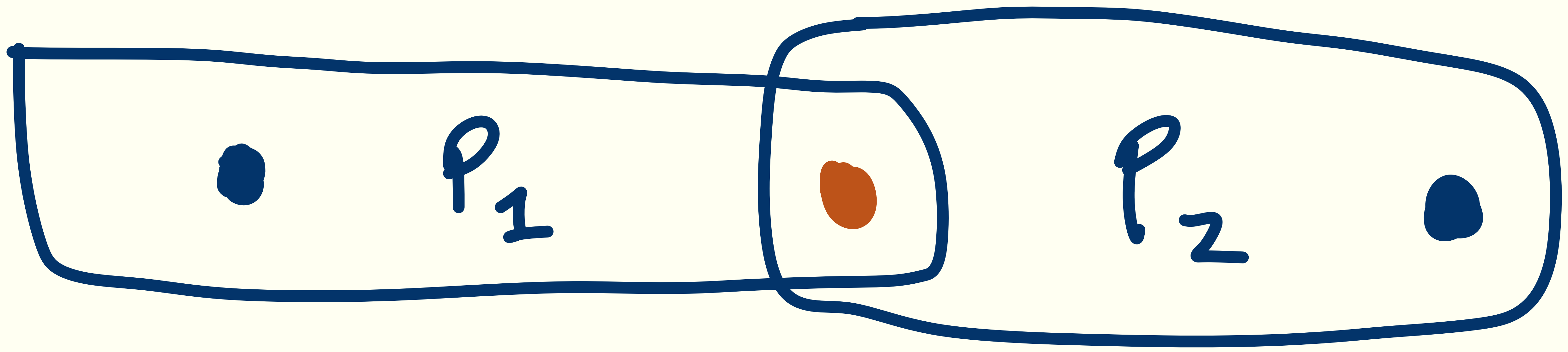


Recipe for solving commuting local Hamiltonians

Let's focus on the 2-local case!



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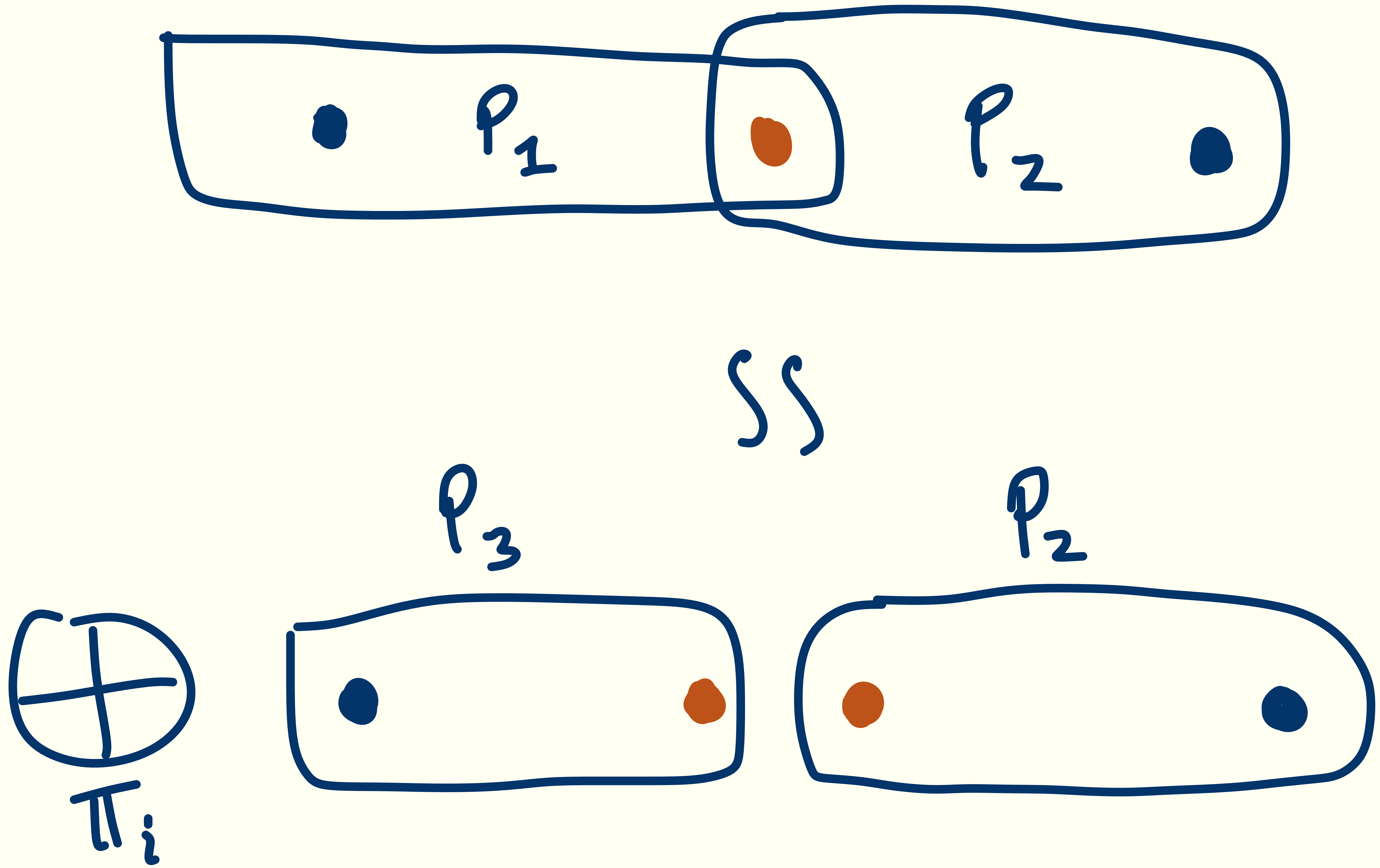

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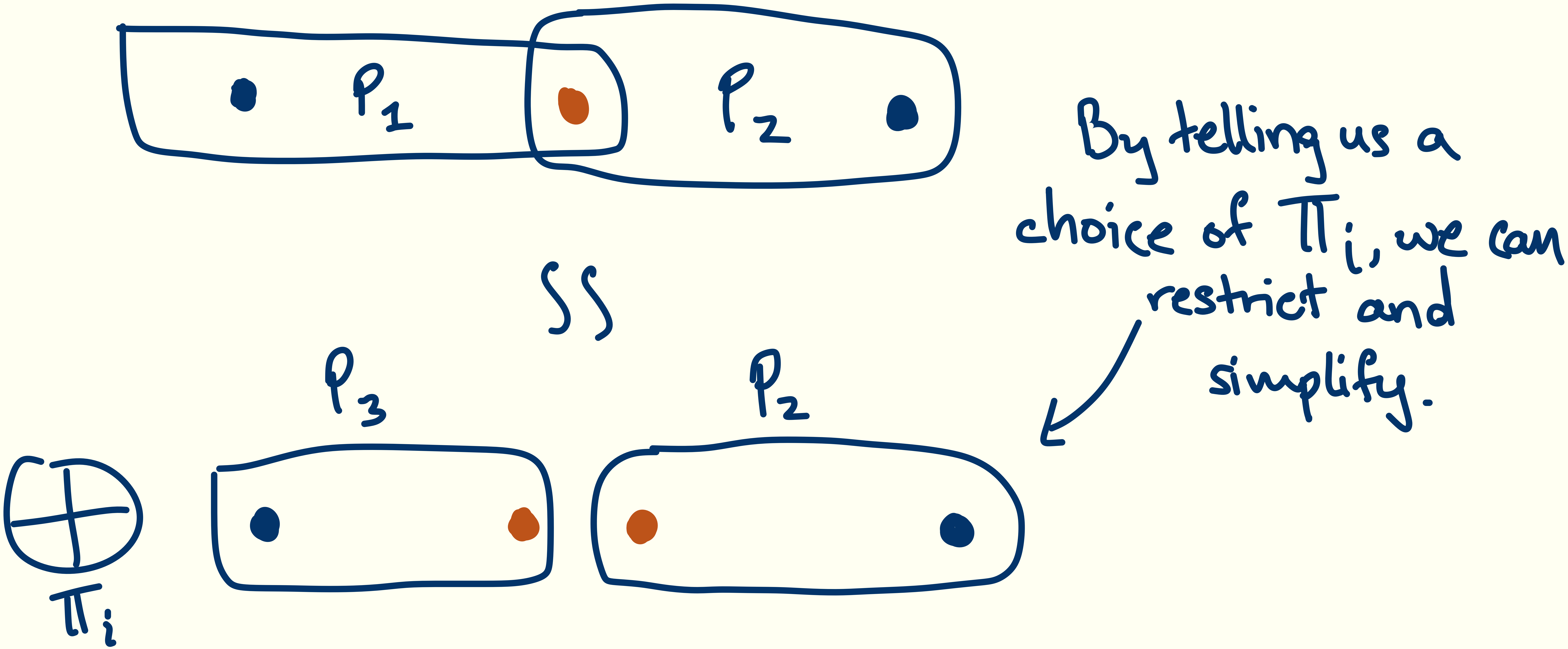
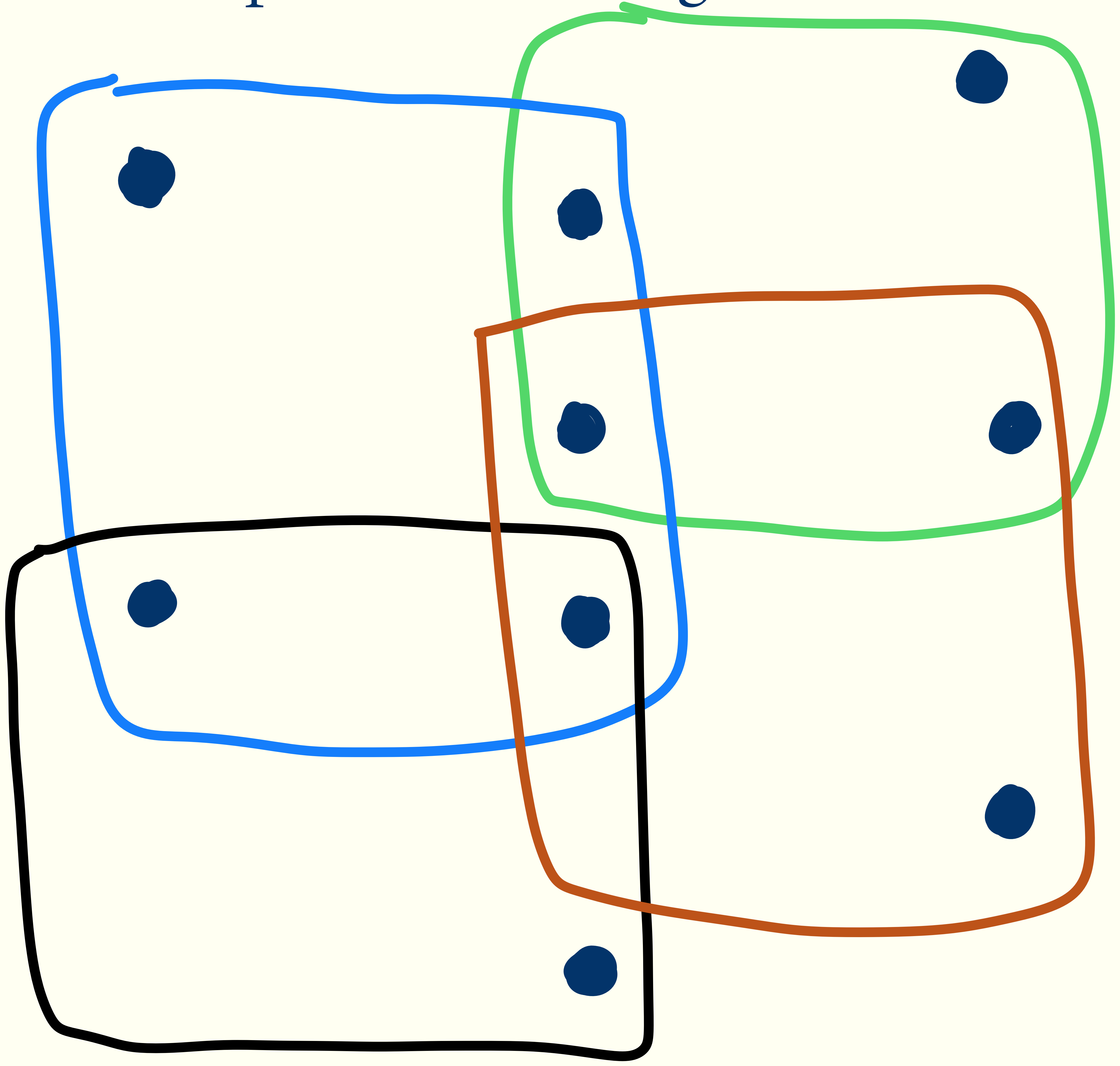


Problems with going beyond 2-local

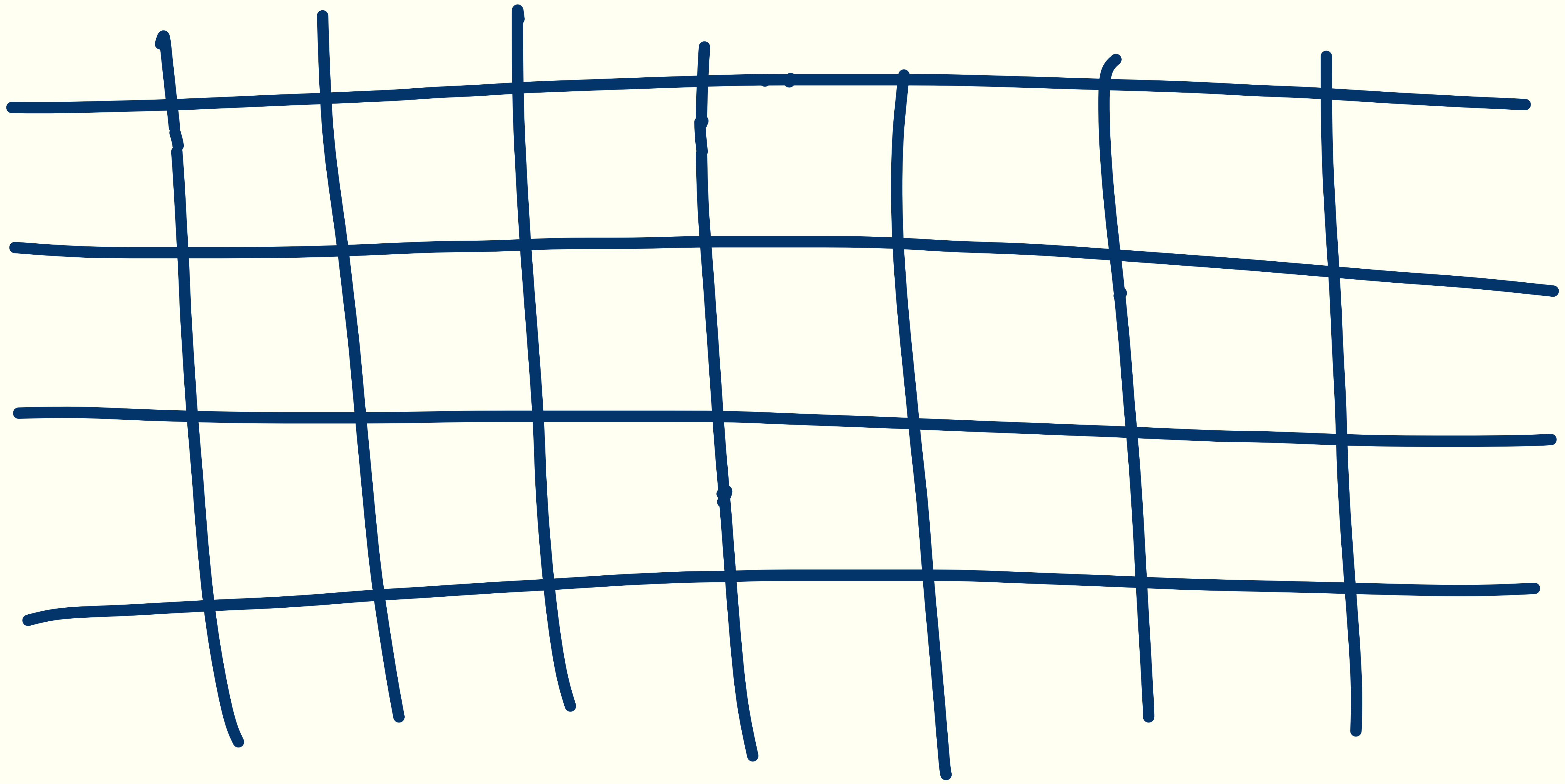
When you have more than 2 terms that interact, the decompositions no longer need to be consistent with each other!

For every intersection, could have different block diagonalizations!



Going beyond 2-local

Instead of using the structure lemma everywhere, maybe we can just try to use it to punch holes in a CLH instance.



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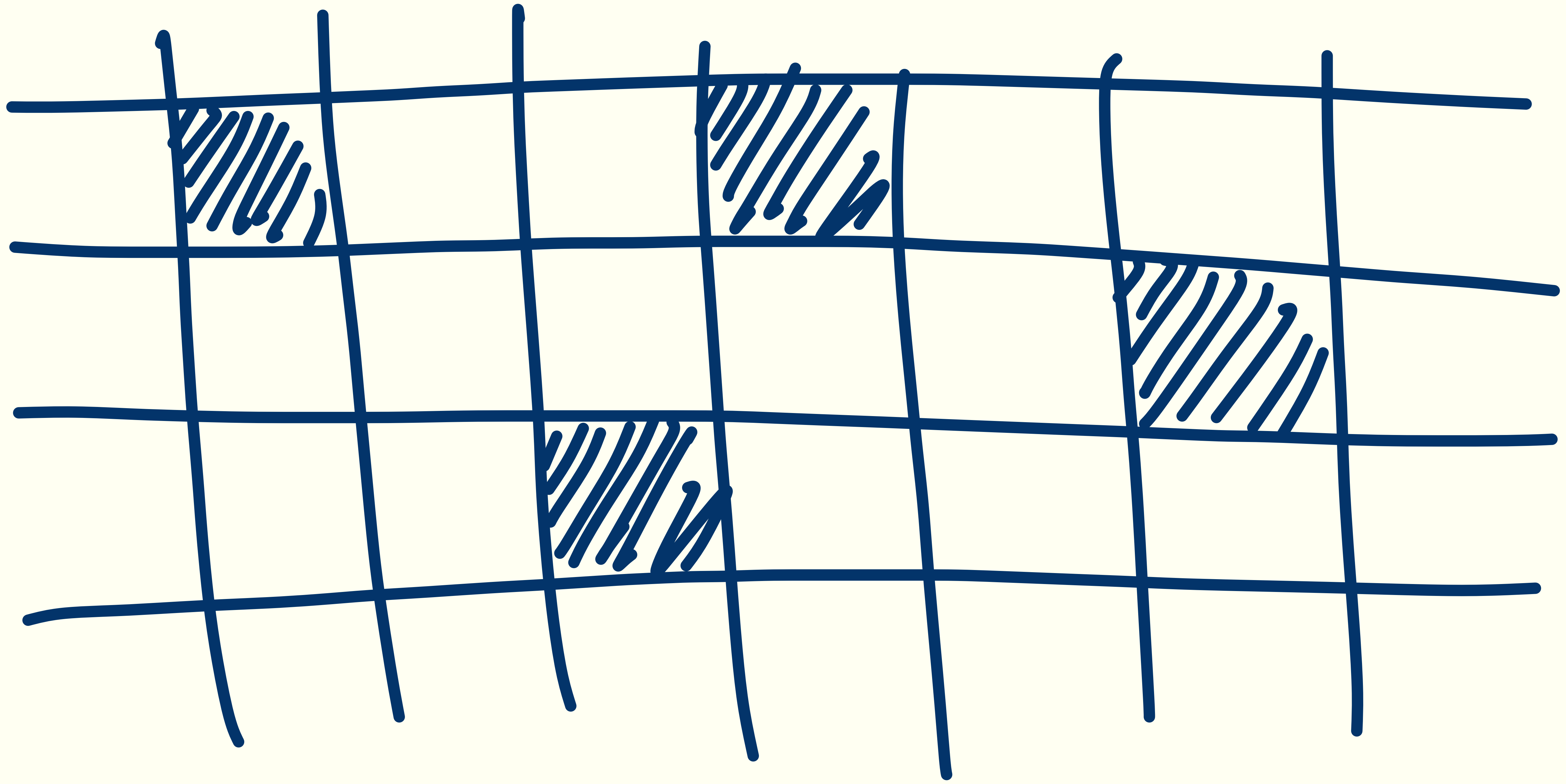

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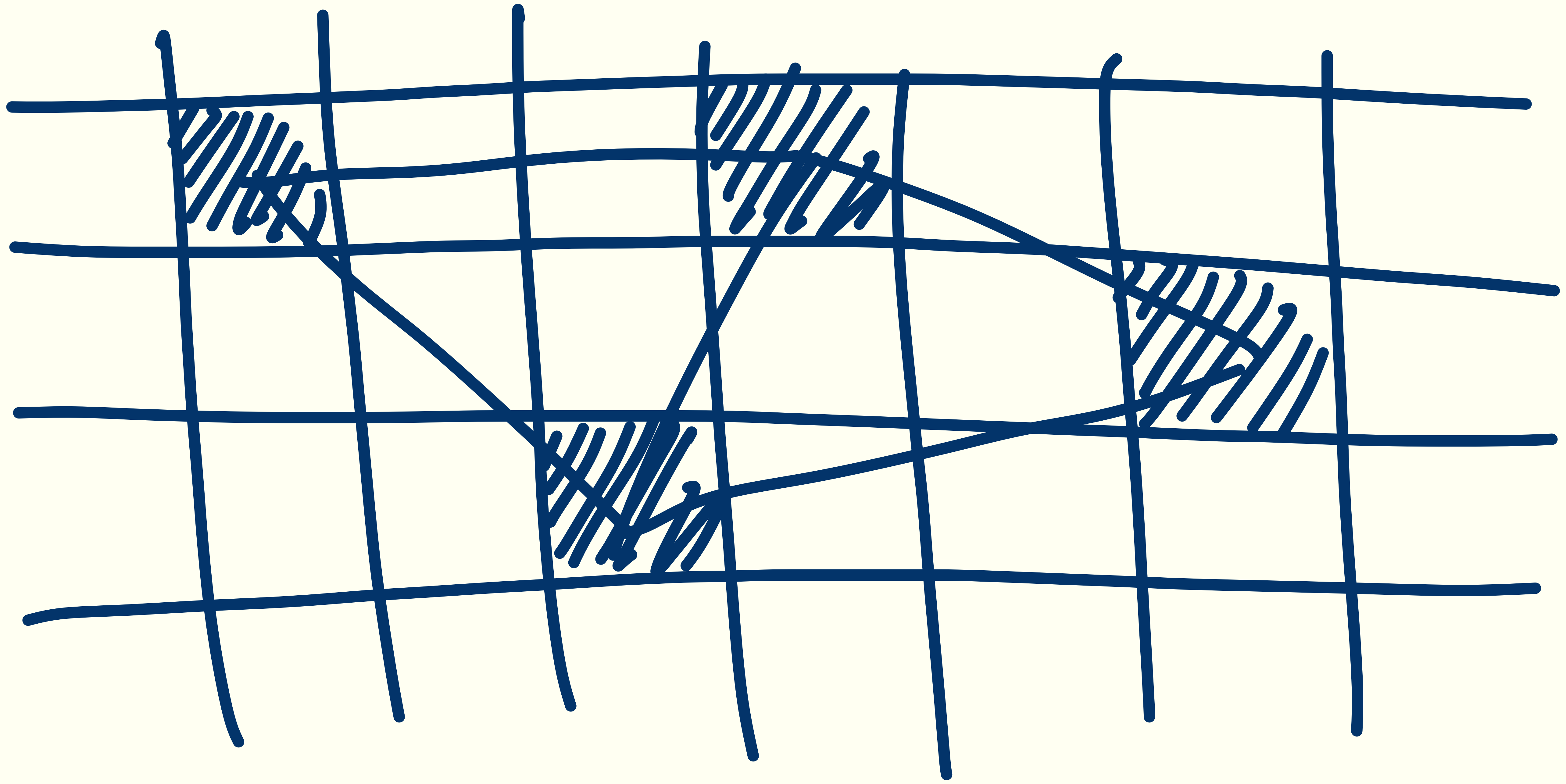

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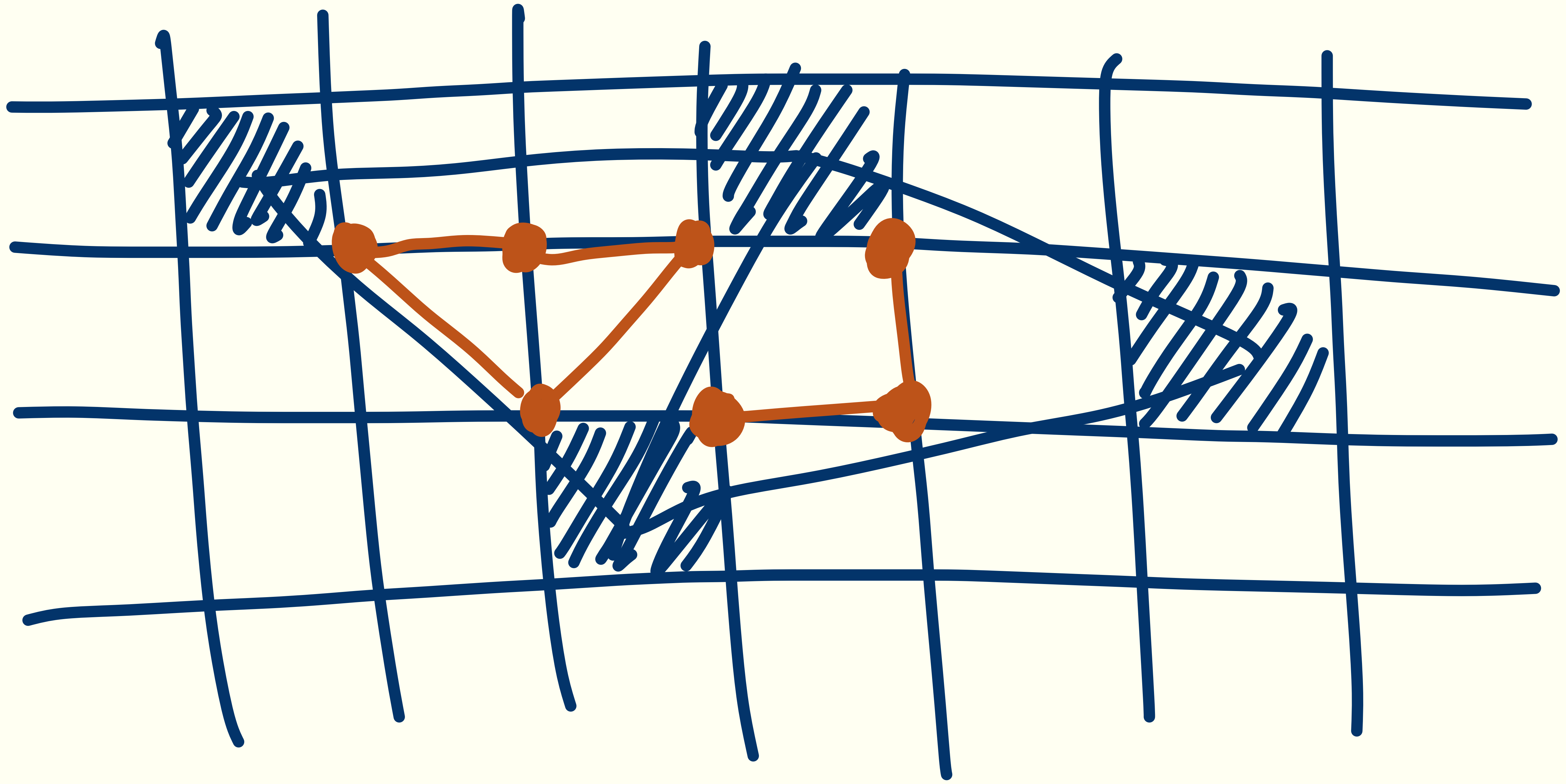

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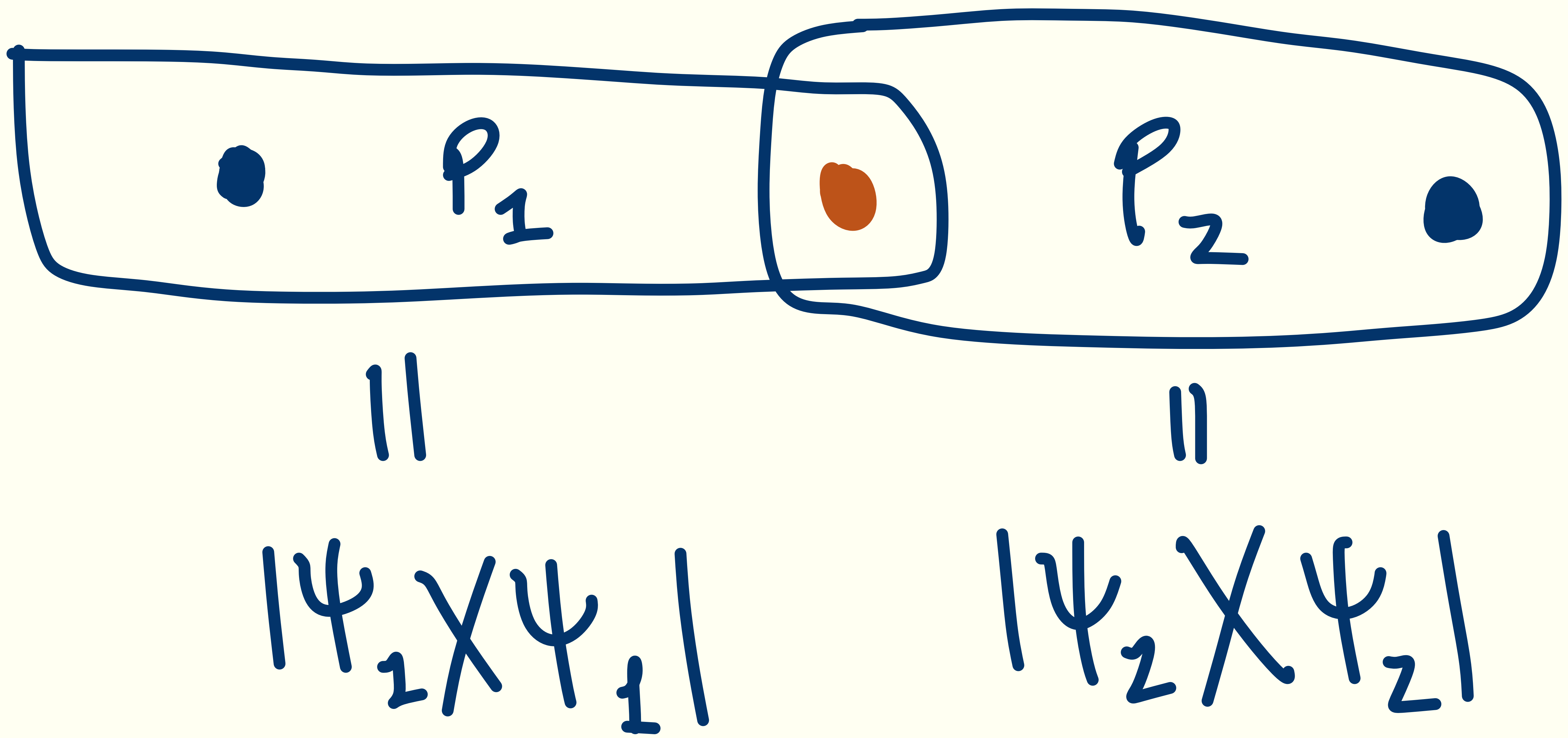
Puncturing a rank-1 Hamiltonian term

Let's consider the case of two rank-1 commuting local Hamiltonian terms.



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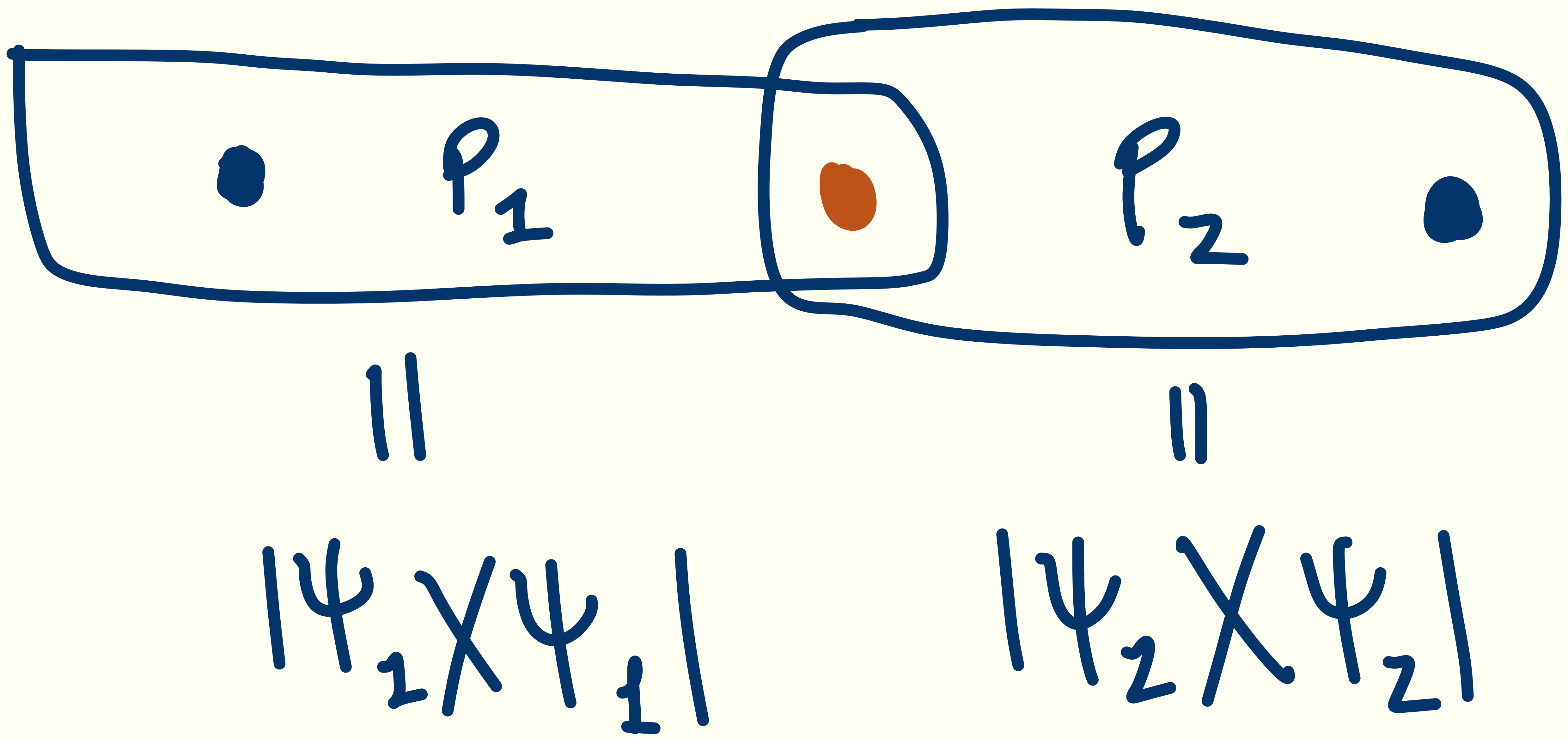

Turns out they decompose into one of the following two cases!

Puncturing a rank-1 Hamiltonian term

Reducing case

$$\langle \psi_1 | \psi_2 \rangle = 0$$

$$\Downarrow$$

On the middle,

$$P_1 \hat{=} \begin{pmatrix} P & \\ & 0 \end{pmatrix}, \quad P_2 \hat{=} \begin{pmatrix} 0 & \\ & Q \end{pmatrix}$$

Singular case

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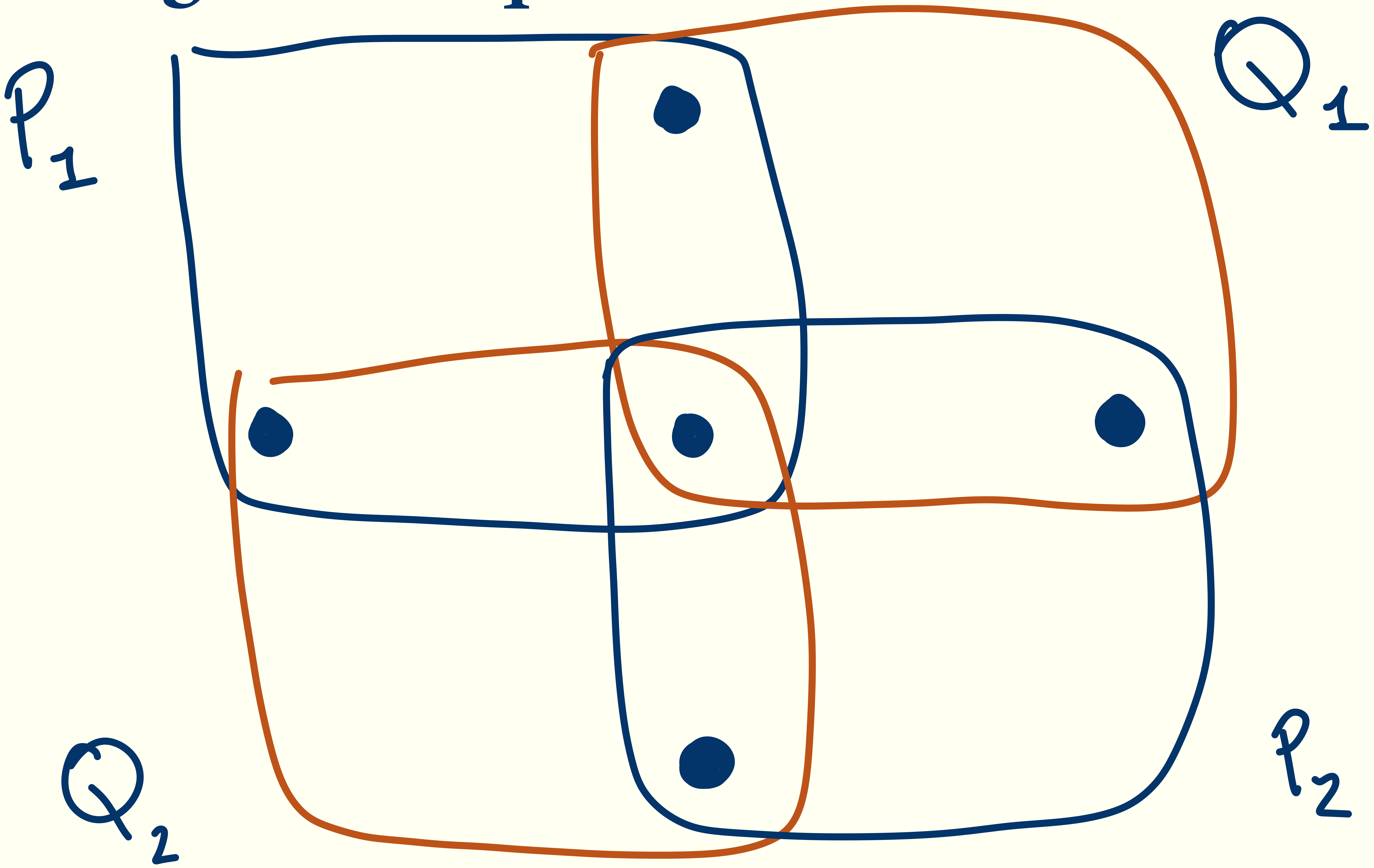
Singular case

$$|\psi_1\rangle = |\phi_1\rangle \otimes |\psi\rangle$$

$$|\psi_2\rangle = |\psi\rangle \otimes |\phi_2\rangle$$

Focusing on the square lattice



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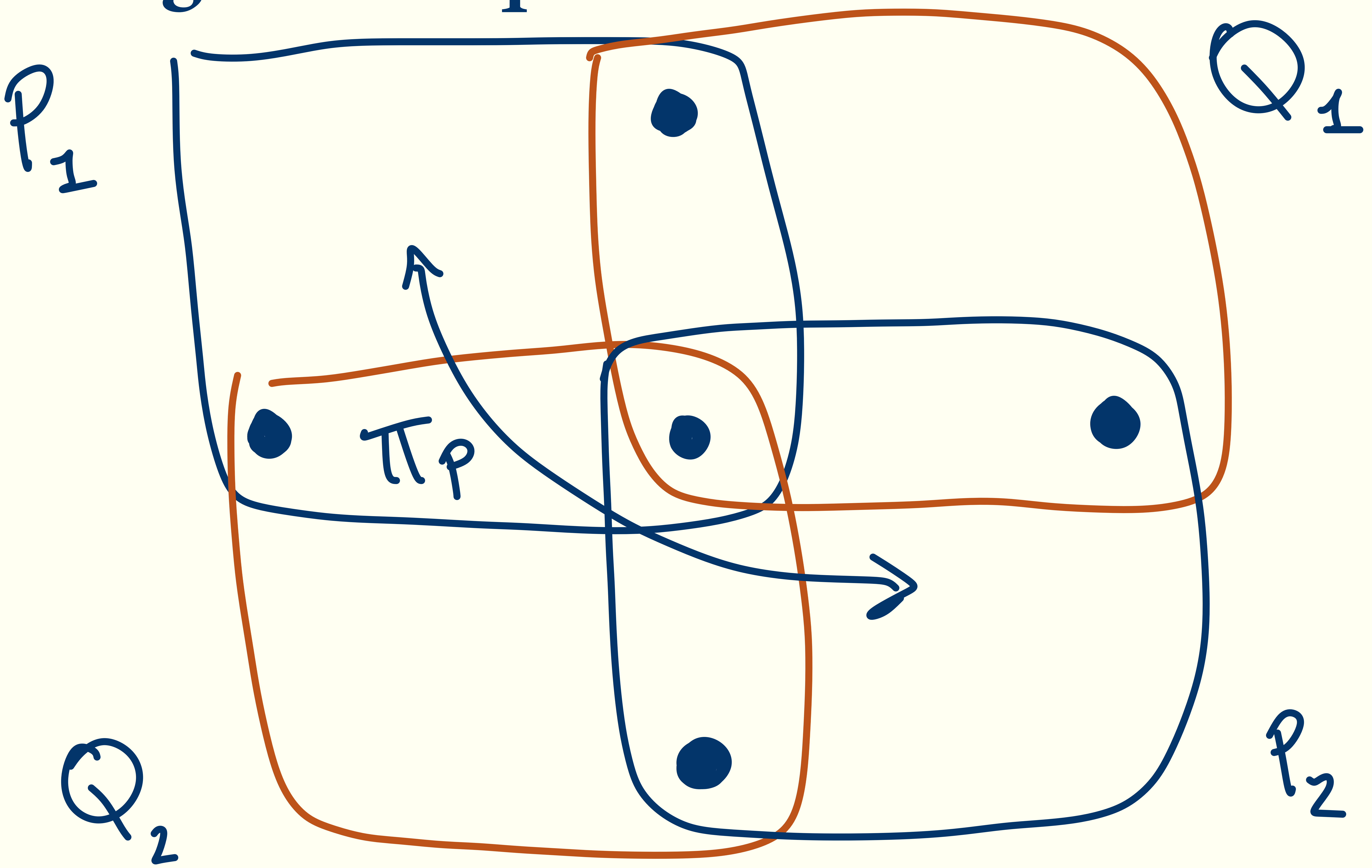

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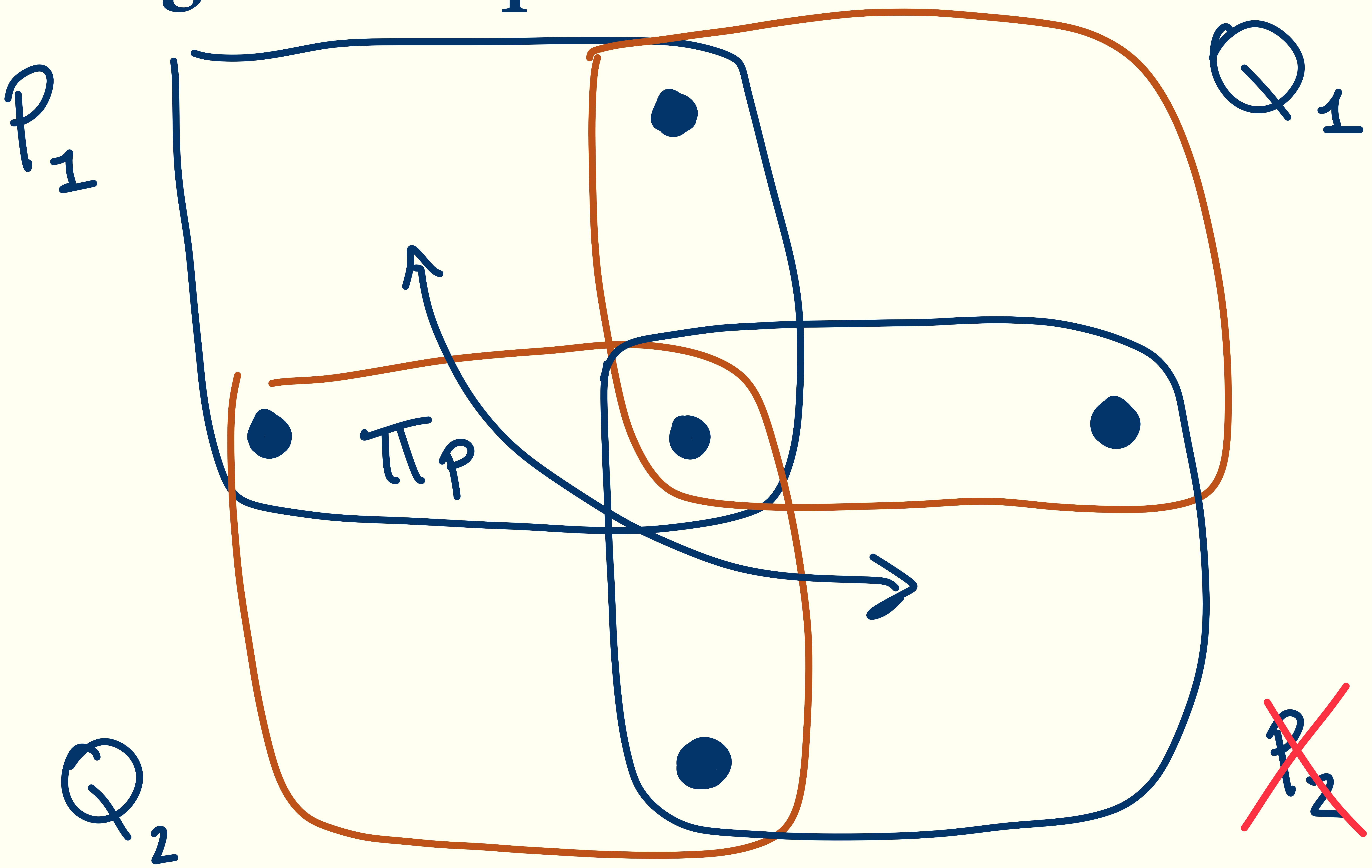

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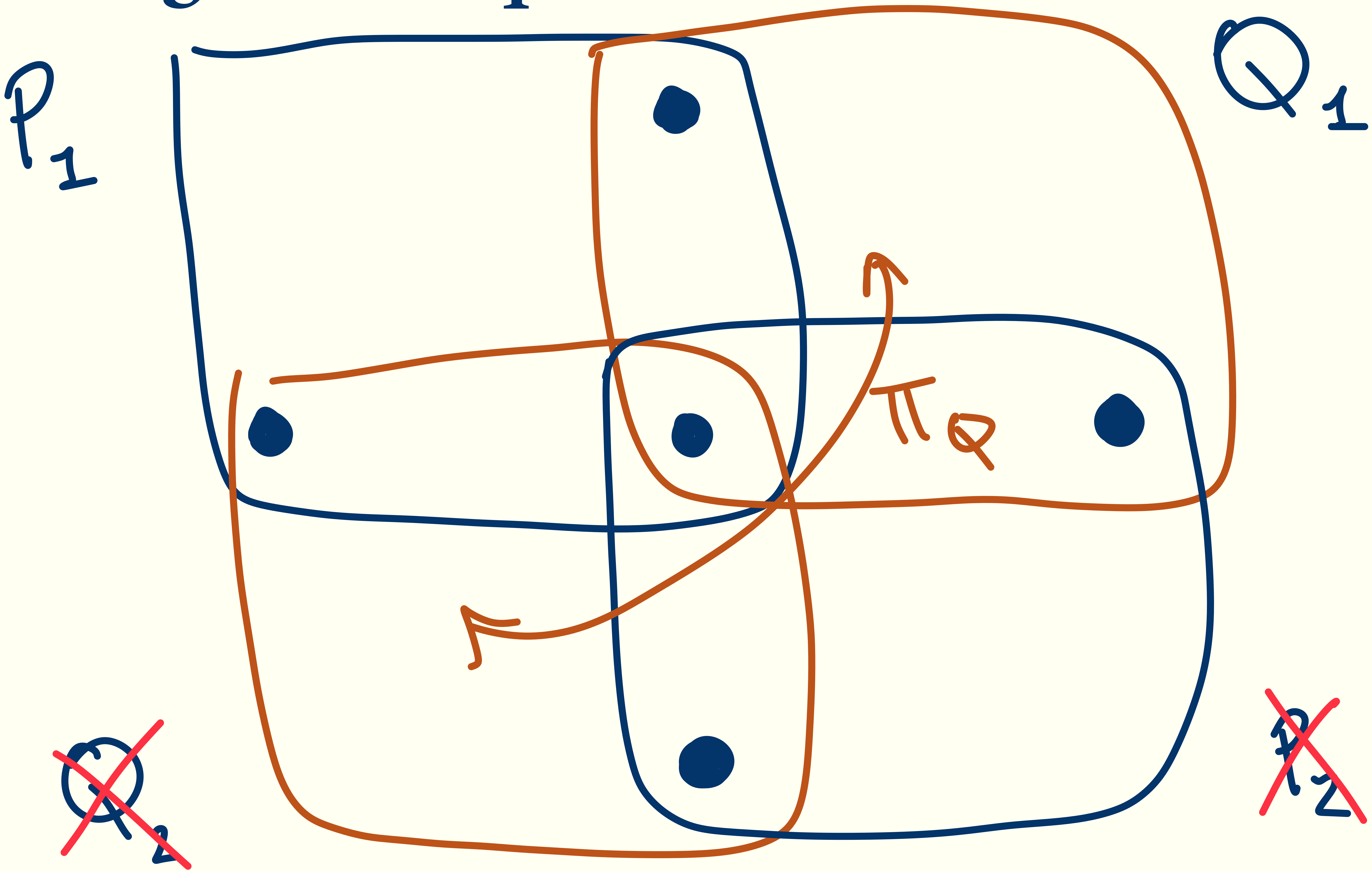
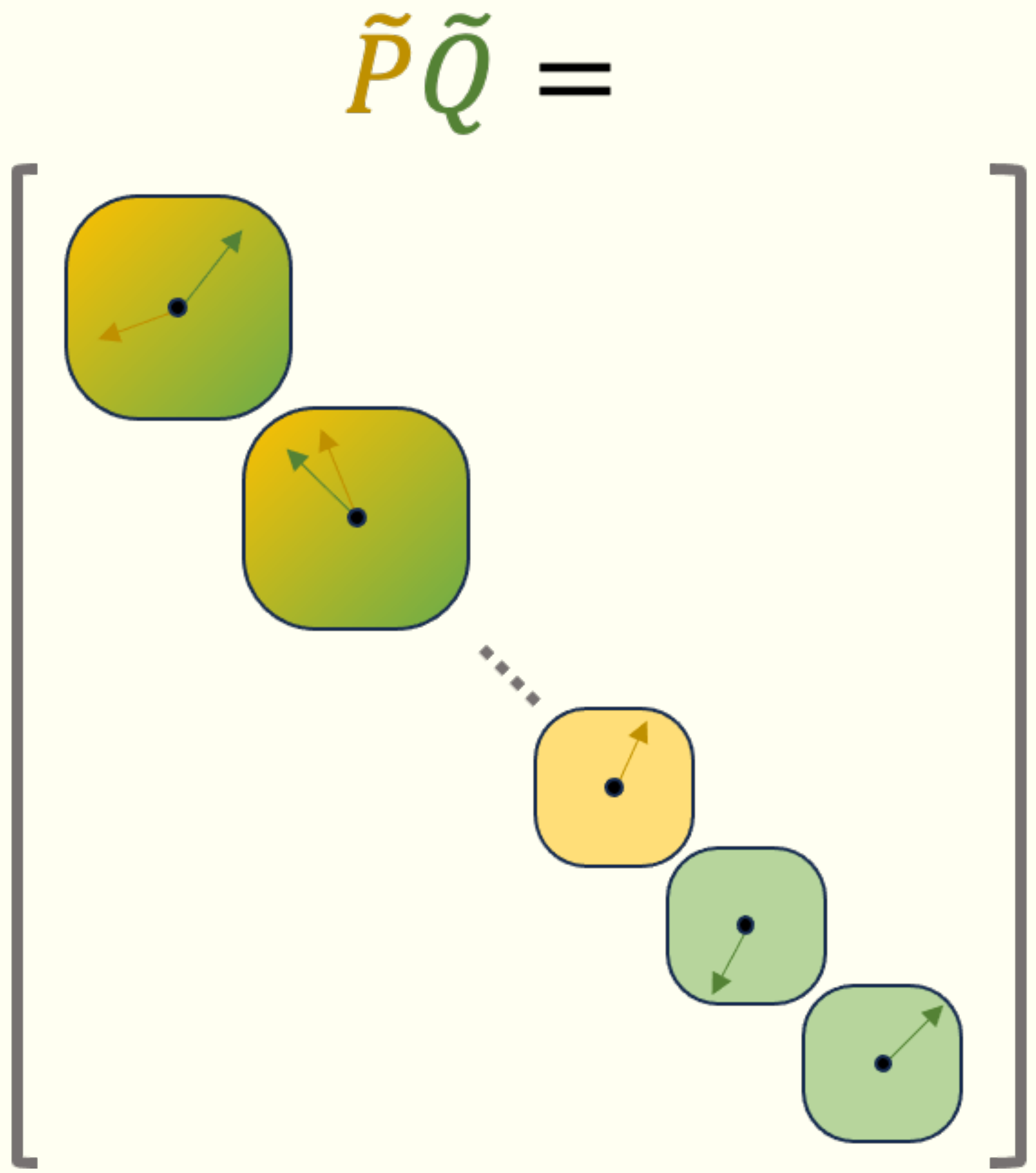


Rounding back to a commuting local Hamiltonian

After puncturing, we are left with a ground space that lives in the intersection of

$$\tilde{P} = \pi_Q - \pi_Q P \pi_Q \quad \text{and} \quad \tilde{Q} = \pi_P - \pi_P Q \pi_P$$

But these no longer commute (because of π), how do we round them?



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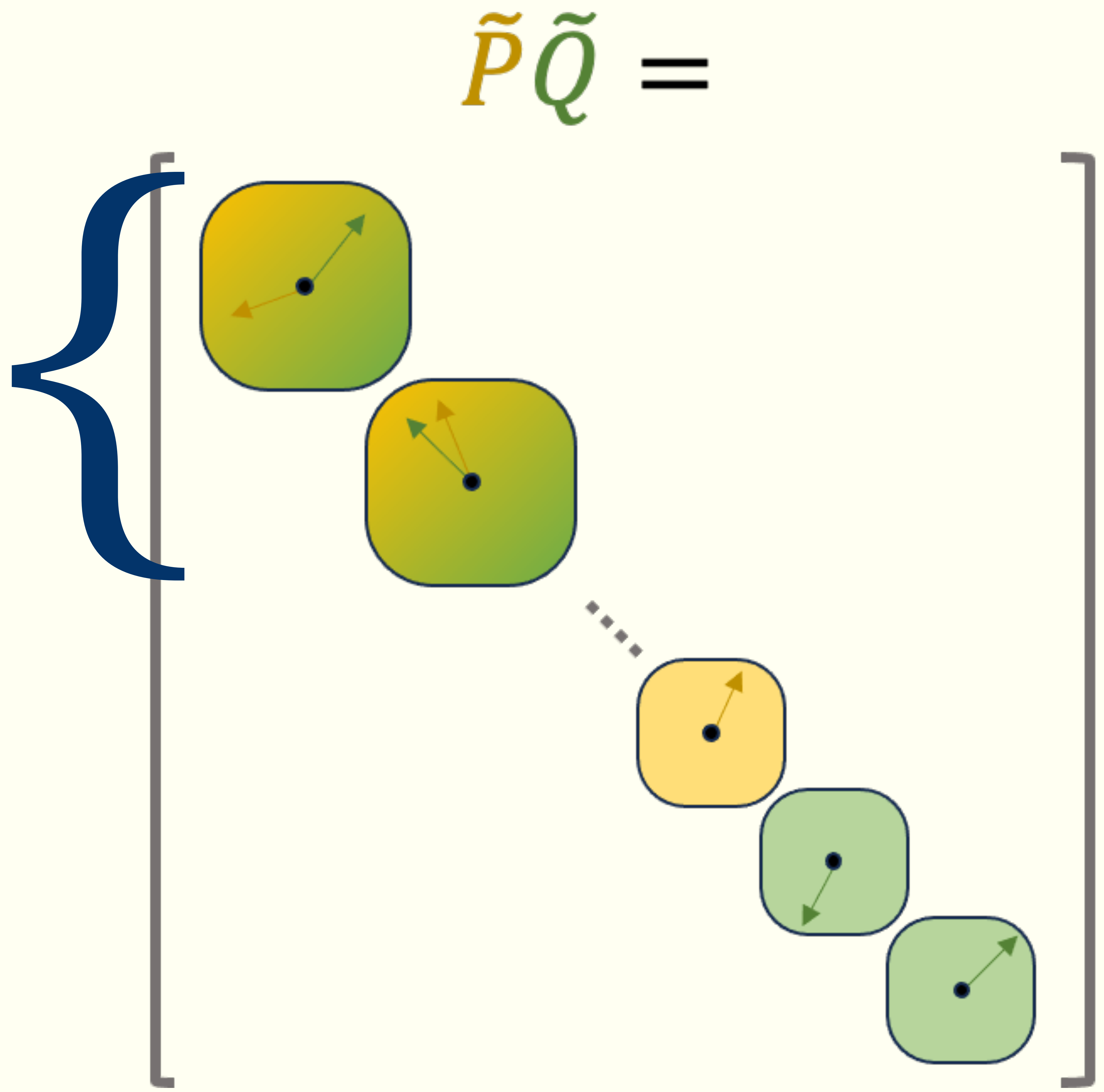

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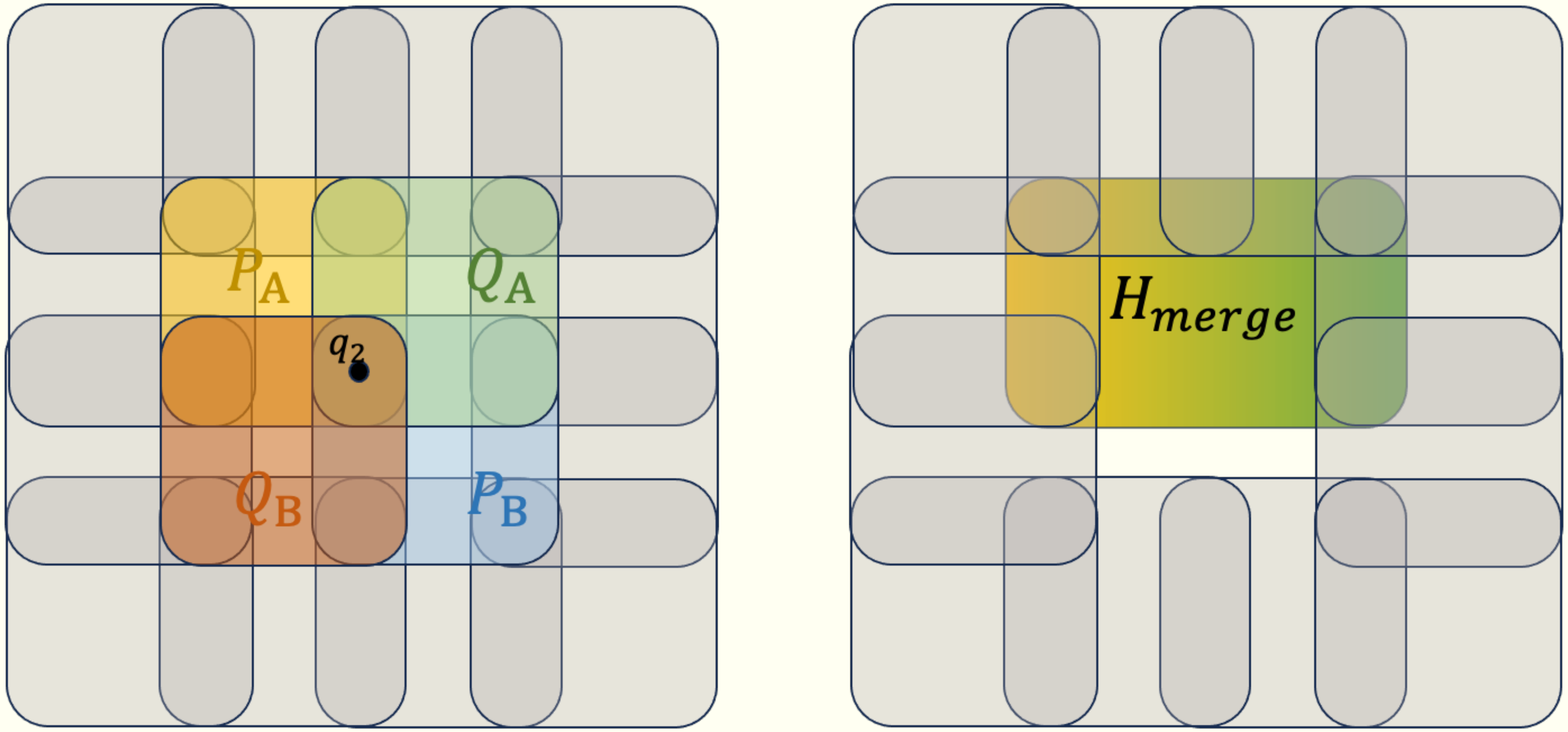
We can use Jordan's lemma to characterize the singular value subspaces and show that they satisfy the original completeness and soundness!

Triangulating a lattice

To reduce the 2D geometrically local (and 3D geometrically local) cases to the 2-local, we punch holes and re-group the Hamiltonian terms.



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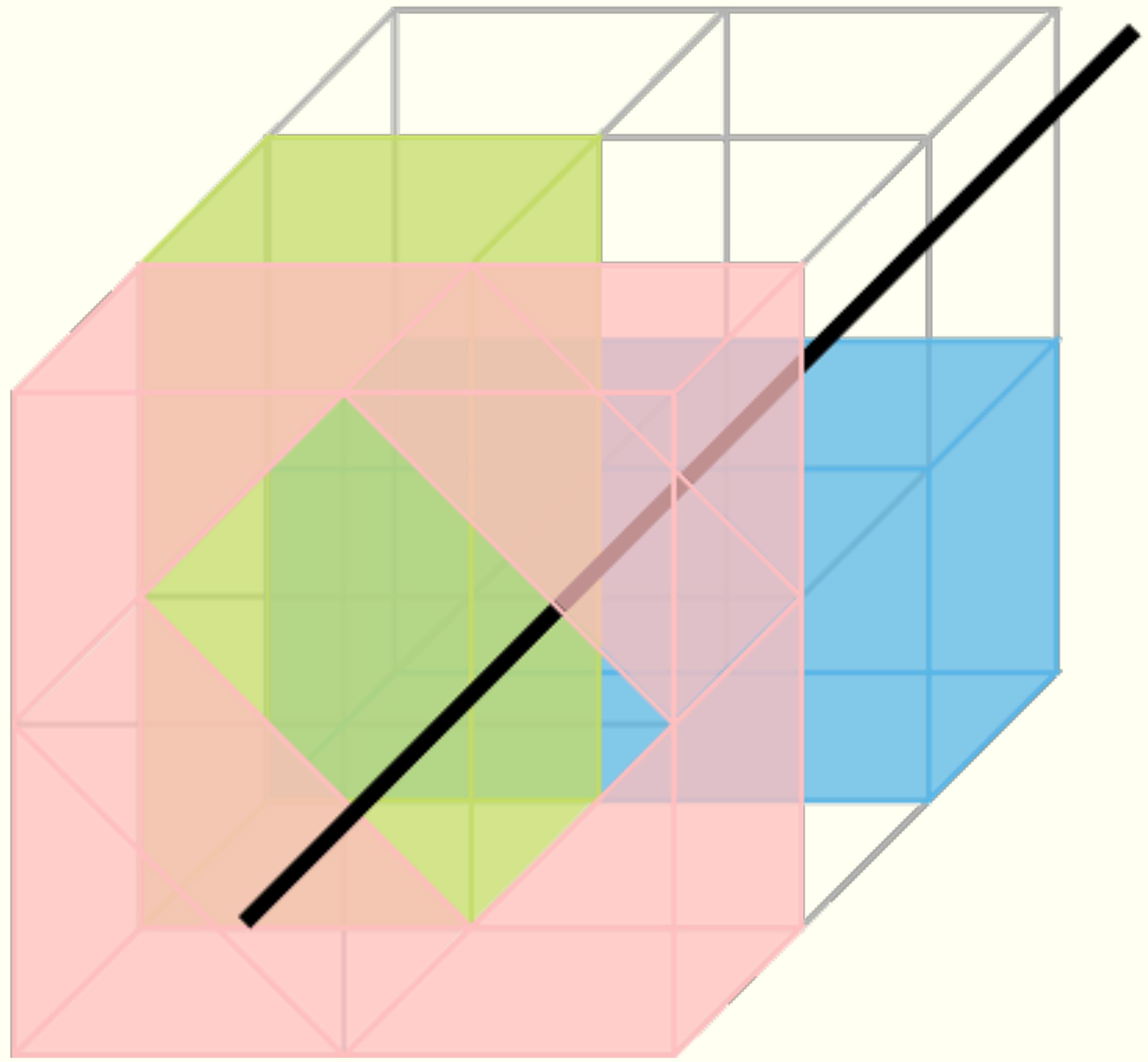

Open questions

- Are rank-1 commuting local Hamiltonians "complete" for the commuting local Hamiltonian problem?
  - For local Hamiltonians and CSP's, they are, but this would imply all 2D geometrically local CLH's are in NP.
- Can we find more tools for understanding commuting local Hamiltonians?
  - Our paper introduced Jordan's lemma as a useful way to round to commuting local Hamiltonians.
- Can we find "gadget" style reductions, instead of relying on the structure lemma?

Thanks for listening!