A General Duality for Representations of Groups

with Applications to Quantum Money, Lightning, and Fire

John Bostanci

Based on joint work with Barak Nehoran and Mark Zhandry

Flipping-Distinguishing duality

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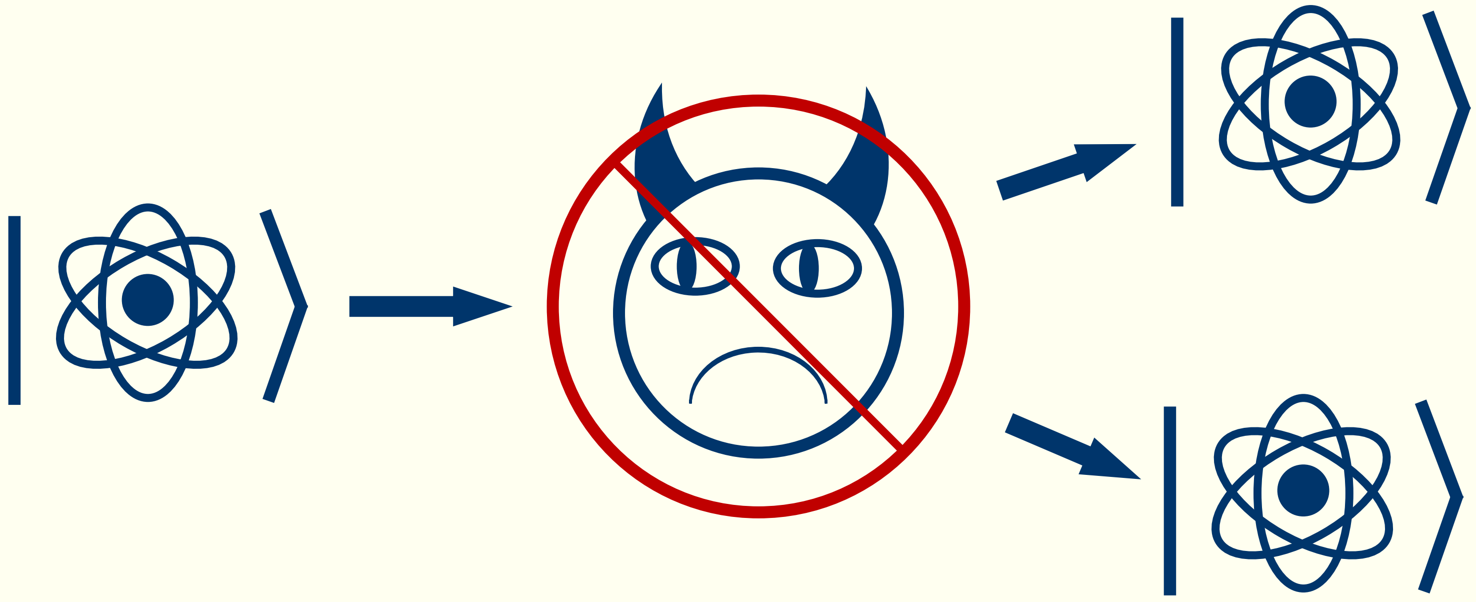
No cloning says no one can clone an **arbitrary** quantum state.



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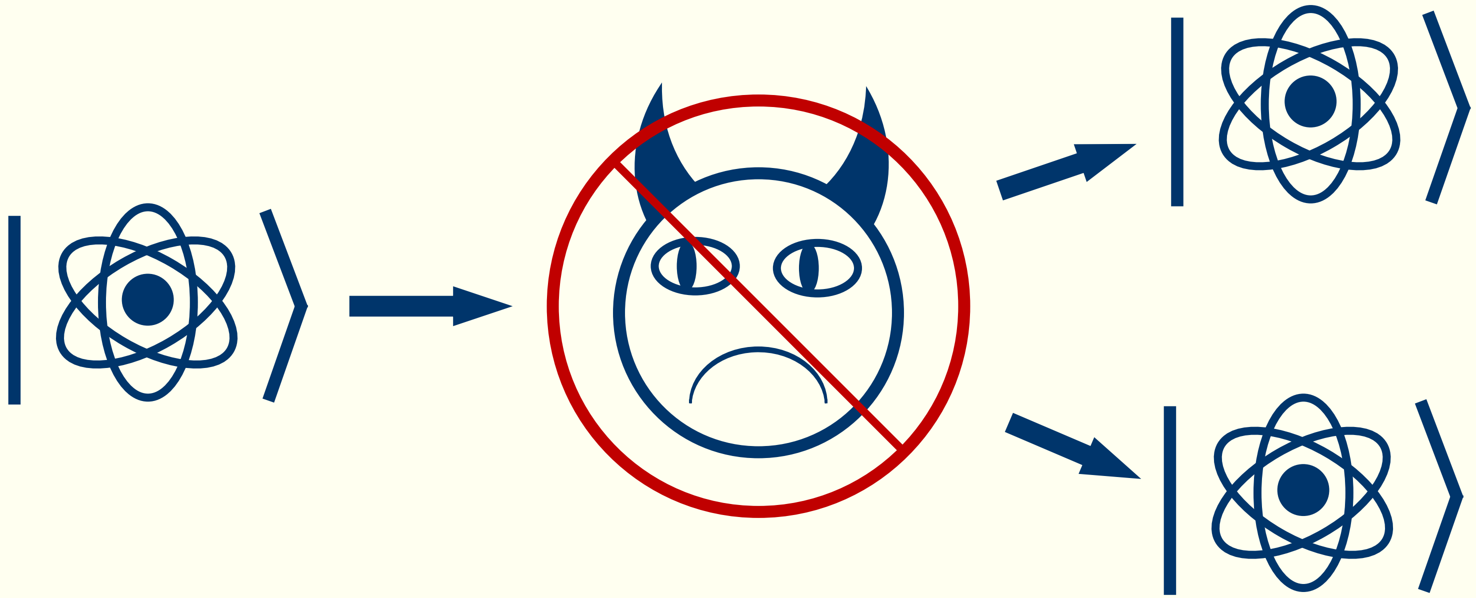

Does this also hold for a family of **useful** quantum states?

From no-cloning to quantum money

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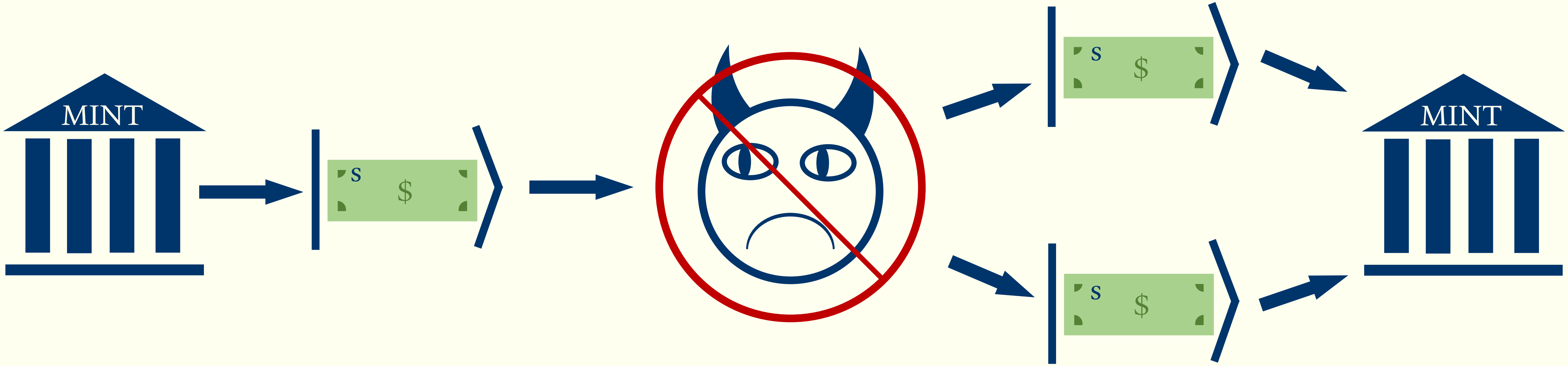

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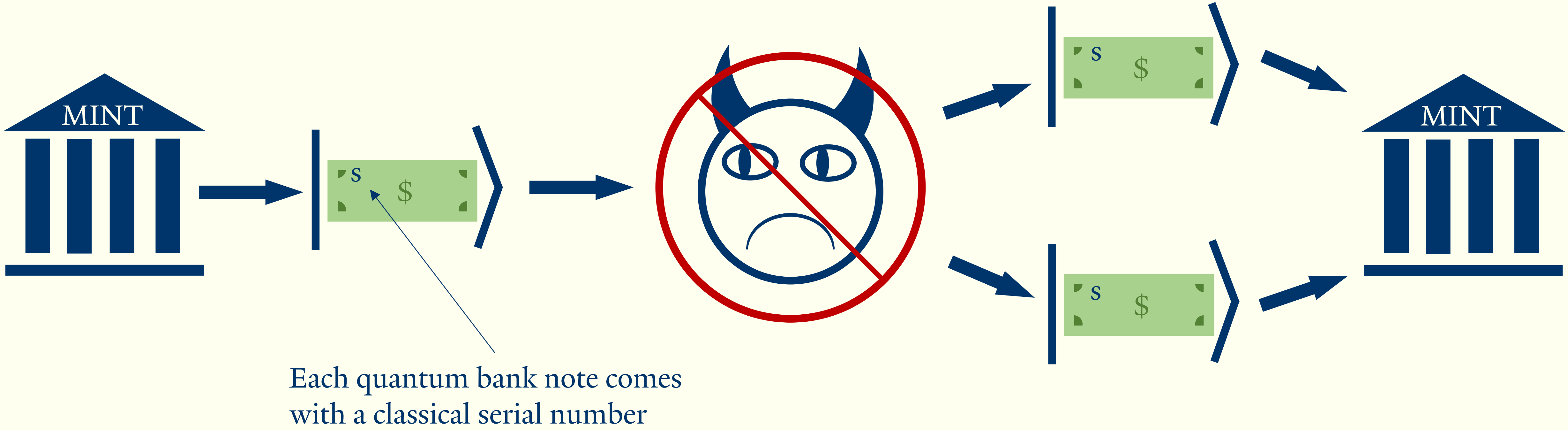

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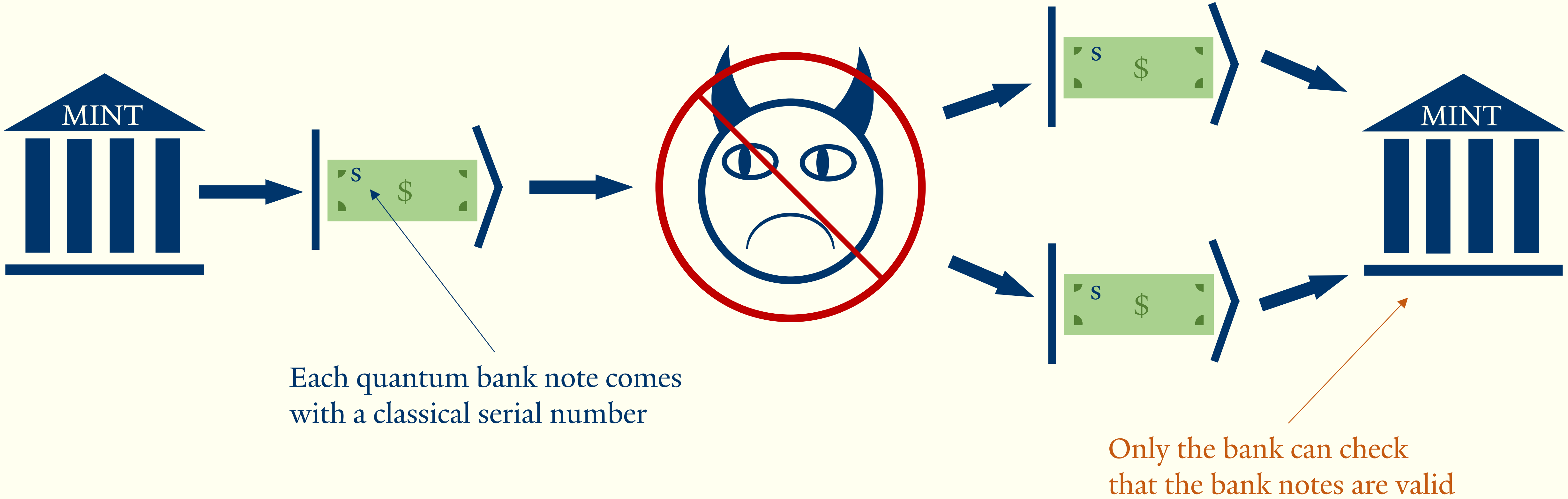
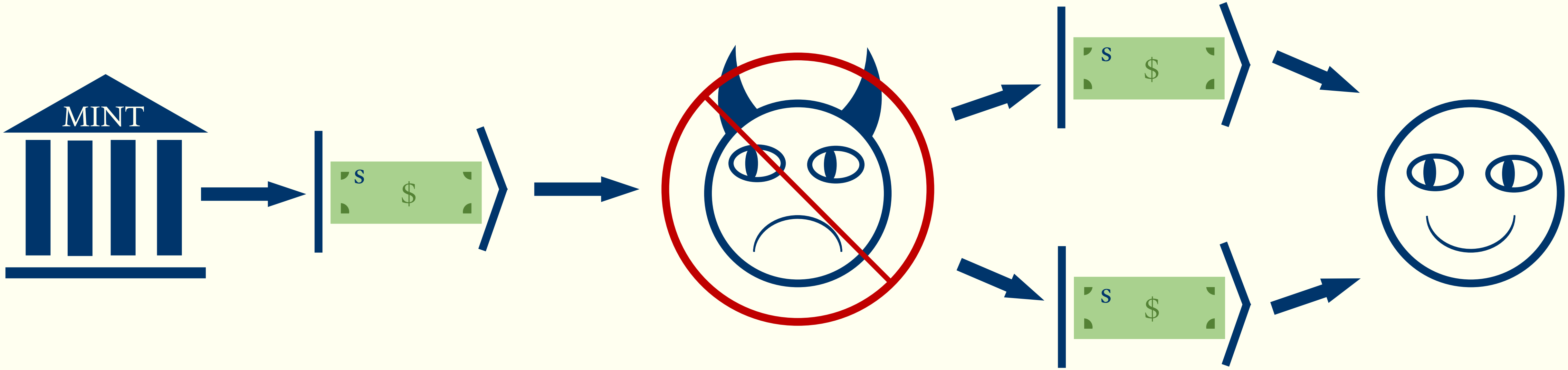


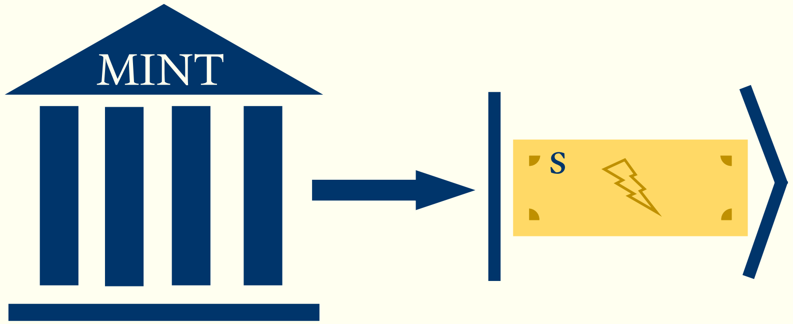
Public-key quantum money

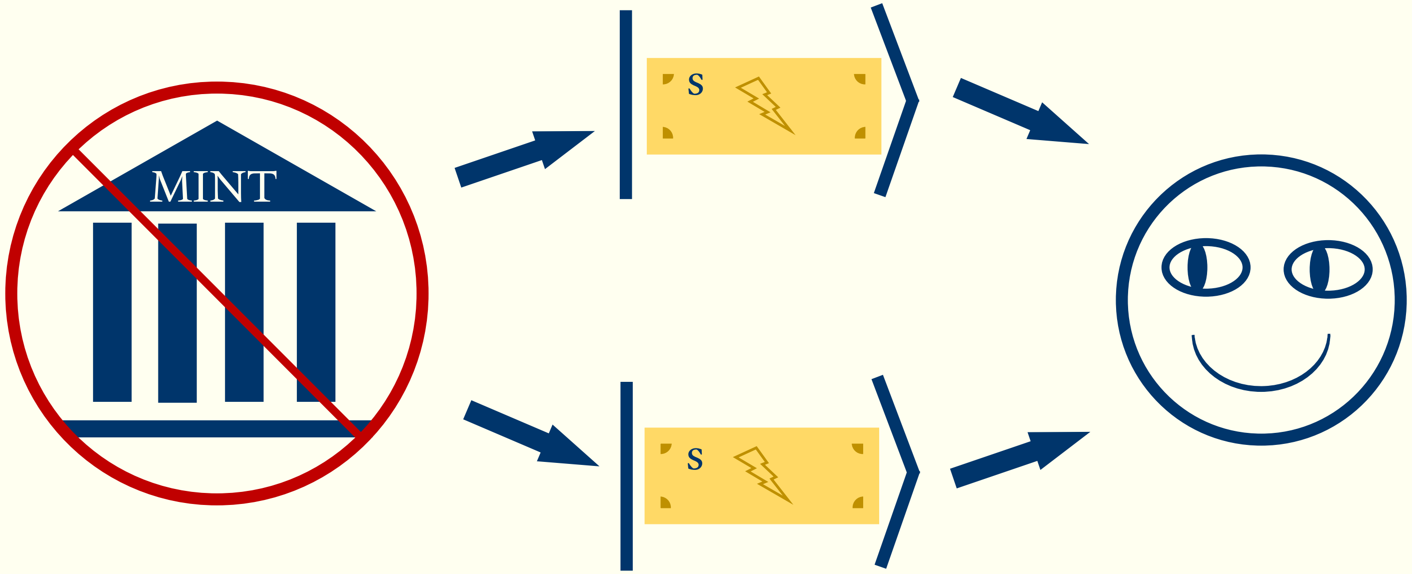
Wiesner, Breidbart, Bennet, Brassard (1982) and Aaronson (2009) proposed quantum money that anyone can verify.



Public-key quantum lightning

Zhandry (2019) formalied a variant of quantum money that is "collision resistant".





Not even the mint can make two notes that have the same serial number!

Unfortunately, constructing quantum money has been really hard!

Only has conjectured security, or completely broken	Security in an idealized model	Security from a well-studied assumption
Aaronson'09 (Random stabilizer states)	Aaronson'09 (Relative to a quantum oracle)	[Zhandry'19]: Post-quantum iO
[Farhi-Gosset-Hassidim-Lutomirski-Shor'10]: knots	[Aaronson-Christiano'12]: classical hidden subspaces oracle	
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Ingredients:

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Invariant subspaces of a group.

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An EPR pair of "group elements"

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Quantum lightning from group actions

To understand the construction, we first need to understand three things:

- Group actions.
- Irreducible representations of groups.
- Quantum Fourier transforms for non-Abelian groups.

Group actions

A group action is a pair of a group G, and set X, a starting element $x \in X$, and an operation

$$*: G \times X \mapsto X$$

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When we say we can implement a group action, we mean we can do:

$$|g\rangle|y\rangle \mapsto |g\rangle|g * y\rangle$$

Representations and irreps

A representation of a group is mapping from a group G to unitary matrices on some vector space V.

$$\mathcal{R} : G \mapsto U(V)$$

What makes it a representation is that it also respects the group action:

$$\mathcal{R}(g)\mathcal{R}(h) = \mathcal{R}(gh)$$

Representations and irreps

Recall that if all of these unitaries commuted, we could simultaneously diagonalize all of them.

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While the quantum Fourier transform from the last slide might seem weird, it has the "usual" form when we consider the left-regular representation:

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For Abelian groups, i, j only go up to 1 and d_λ is 1 for all irreps.

Quantum lightning from group actions

In the construction, we'll need to start with a group action for a group that has an **efficient quantum Fourier transform**, e.g.

1. Any group whose size doesn't scale in n.
2. Dihedral group.
3. Symmetric group.

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We only need the irrep label.

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Our candidate problem is called "dirty fixed point testing".

Dirty fixed point testing and security

Simplified setup for dirty fixed point testing:

1. An “extraction” unitary, Extract
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2. A state $|\psi\rangle$ such that $\text{Extract} \cdot |\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$.
3. Two operators L and R such that:

$$\text{Extract} \cdot L\,|\psi\rangle = |\phi_1\rangle \otimes |\phi_2'\rangle, \text{ and}$$
$$\text{Extract} \cdot R\,|\psi\rangle \text{ is far from } |\phi_1\rangle \otimes \text{id}.$$

Question: Determine if a challenger is applying L or R.

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Dirty fixed point testing is definitely easy with two copies of $|\psi\rangle$:

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Now we need to find hard instances!

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Preaction security:

It's hard to distinguish between a challenger that applies a random action, versus a challenger that applies a random action and a random preaction.

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Implementing the Extract becomes Fourier extraction.

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We show that the following tasks are computationally equivalent:

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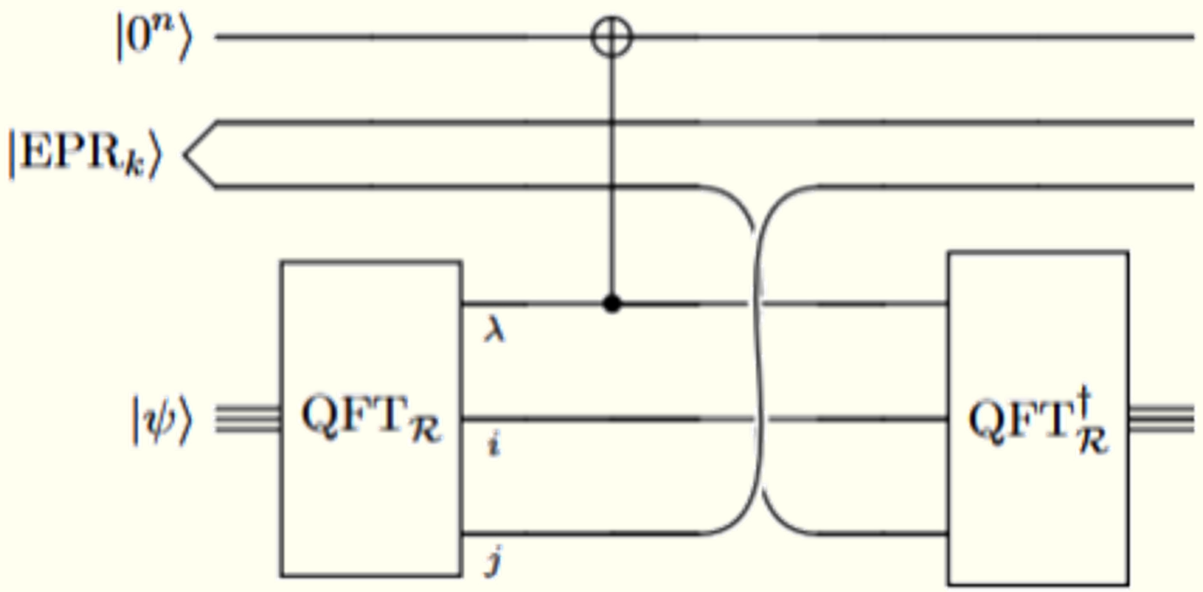

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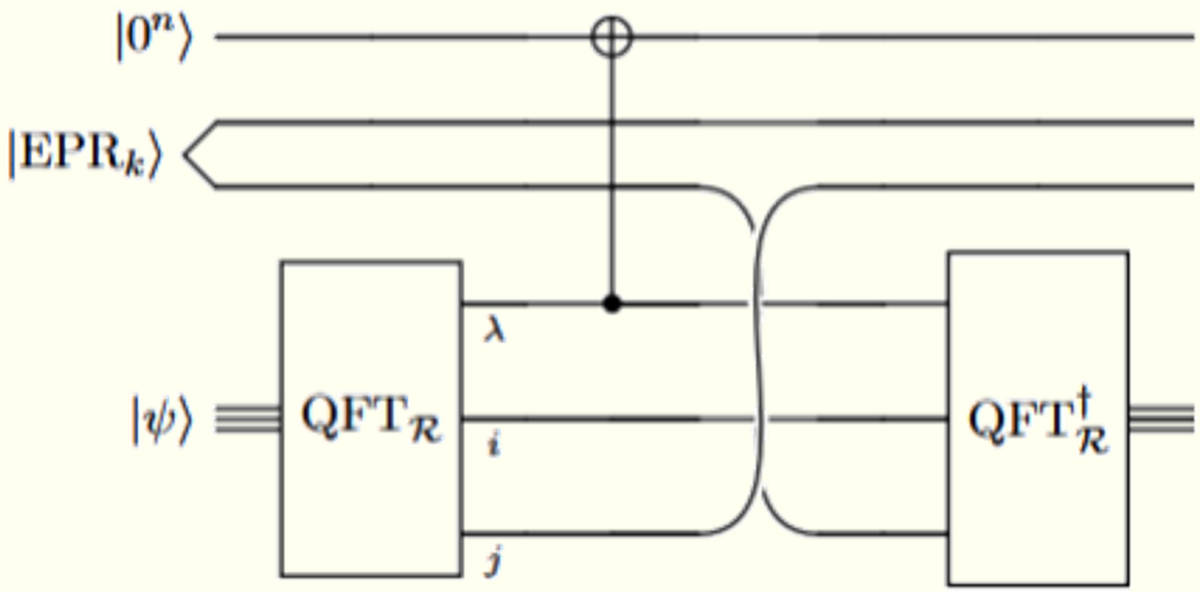

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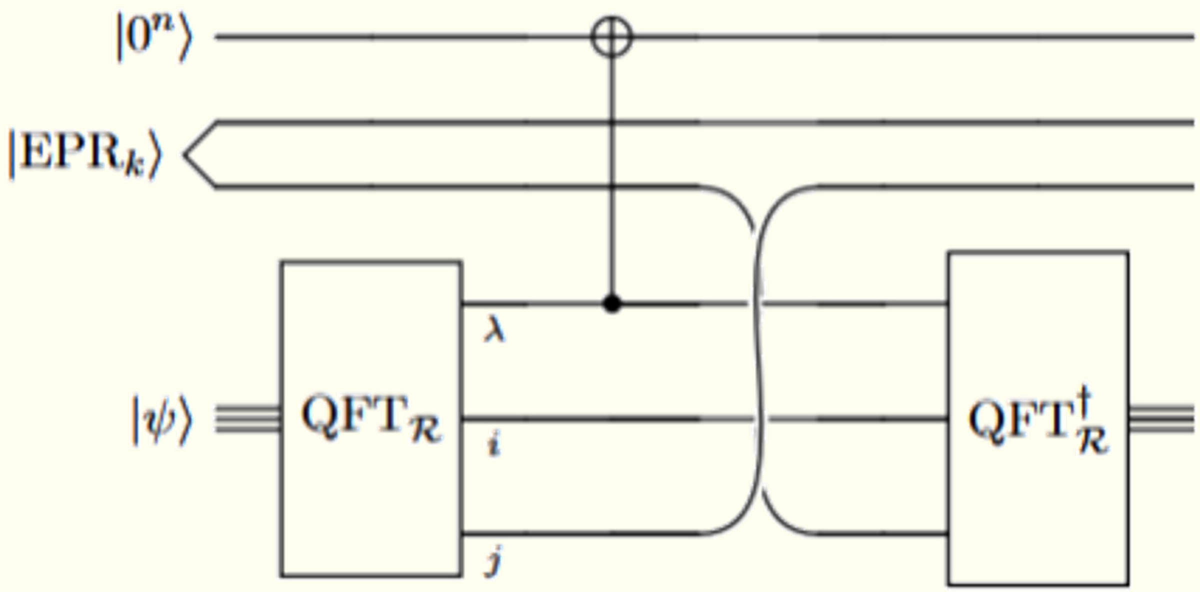

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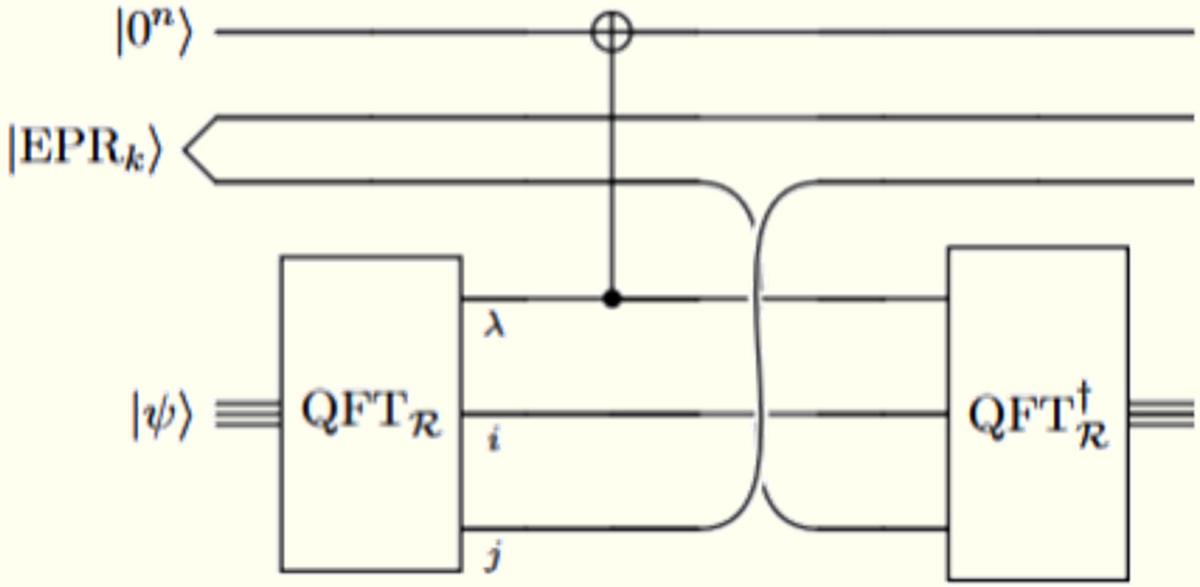

λ is measured

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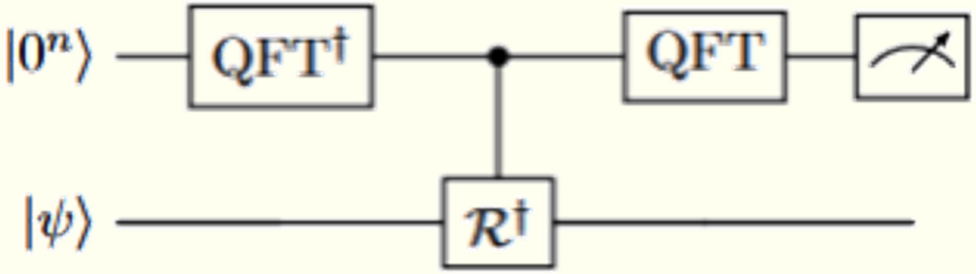
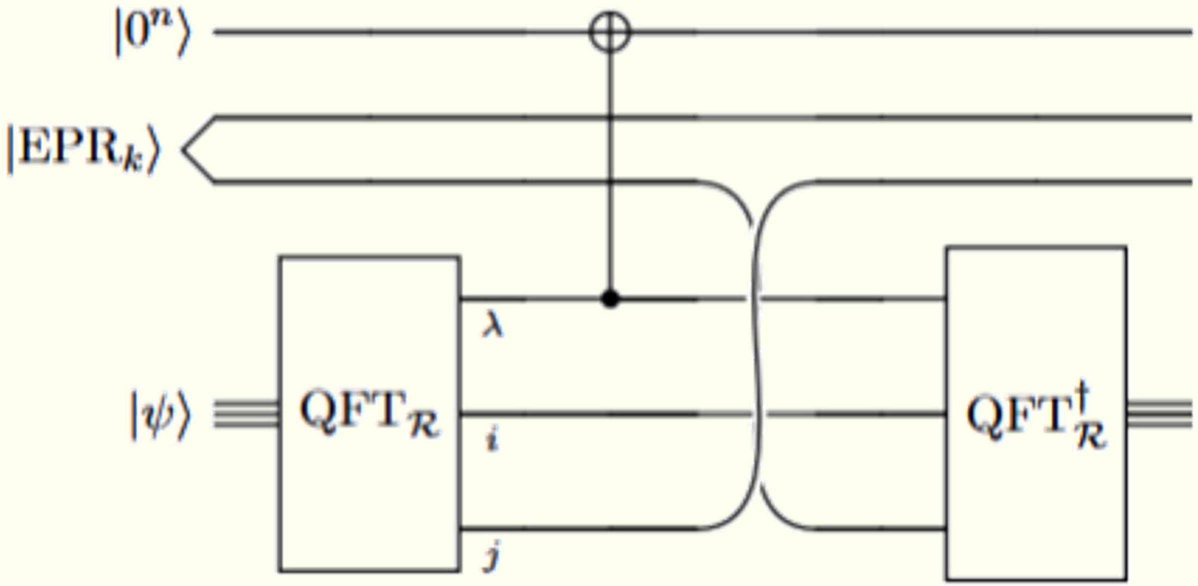
Rest of the state depends only on i

Implementing Fourier extraction

Turns out, the following simple circuit implements Fourier extraction:



$\approx$



Open questions

- Can you reduce preaction security to a "standard" assumption, like discrete log being hard, or the hidden subgroup problem being hard?
- Can you build other things from preaction secure group actions? For example, one-shot signatures, or copy-protected software?
- Can we find a falsifiable variant of preaction indistinguishability? For example, if the group action had a trapdoor that allowed the challenger to implement a random preaction.