A General Quantum Duality for Representations of Groups

and Applications to Quantum Money, Lightning, and Fire

John Bostanci

Based on joint work with Barak Nehoran and Mark Zhandry

Discovery Fiction: Quantum Lightning from Abelian Group Actions

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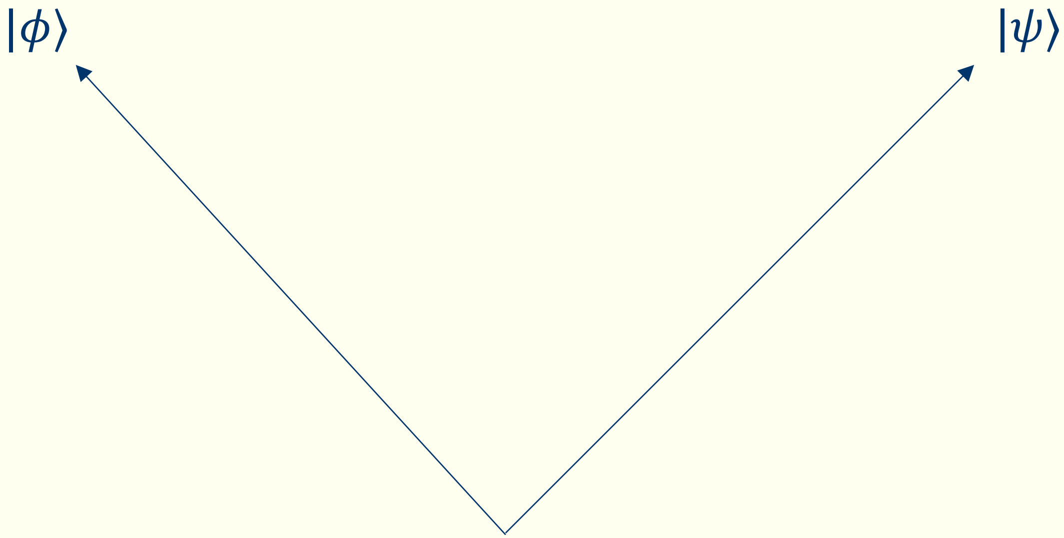
- However, restricting to Abelian groups meant that the security proof required a black-box assumption, and complicated the scheme.

- We hoped that generalizing the construction to non-Abelian groups might fix these problems, but it's not that easy.
- Along the way, identified an interesting algorithmic task concerning representations of groups.

The General Quantum Duality Theorem for Representations of Groups

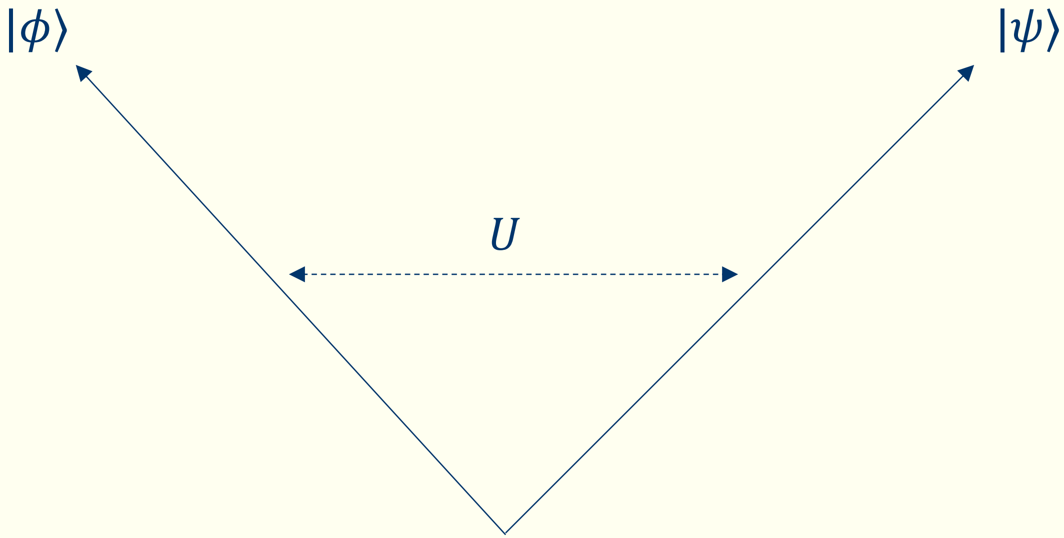
Swapping-distinguishing duality

Imagine we have two orthogonal states, $|\psi\rangle$ and $|\phi\rangle$.



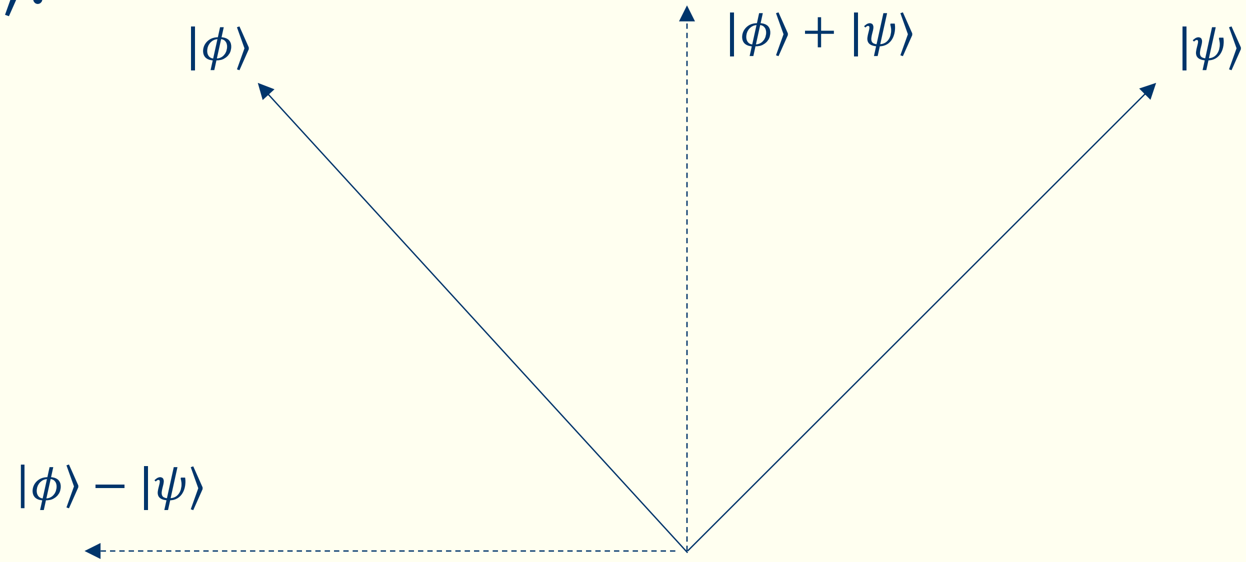
Swapping-distinguishing duality

How hard is it to (approximately) swap $|\psi\rangle \leftrightarrow |\phi\rangle$?



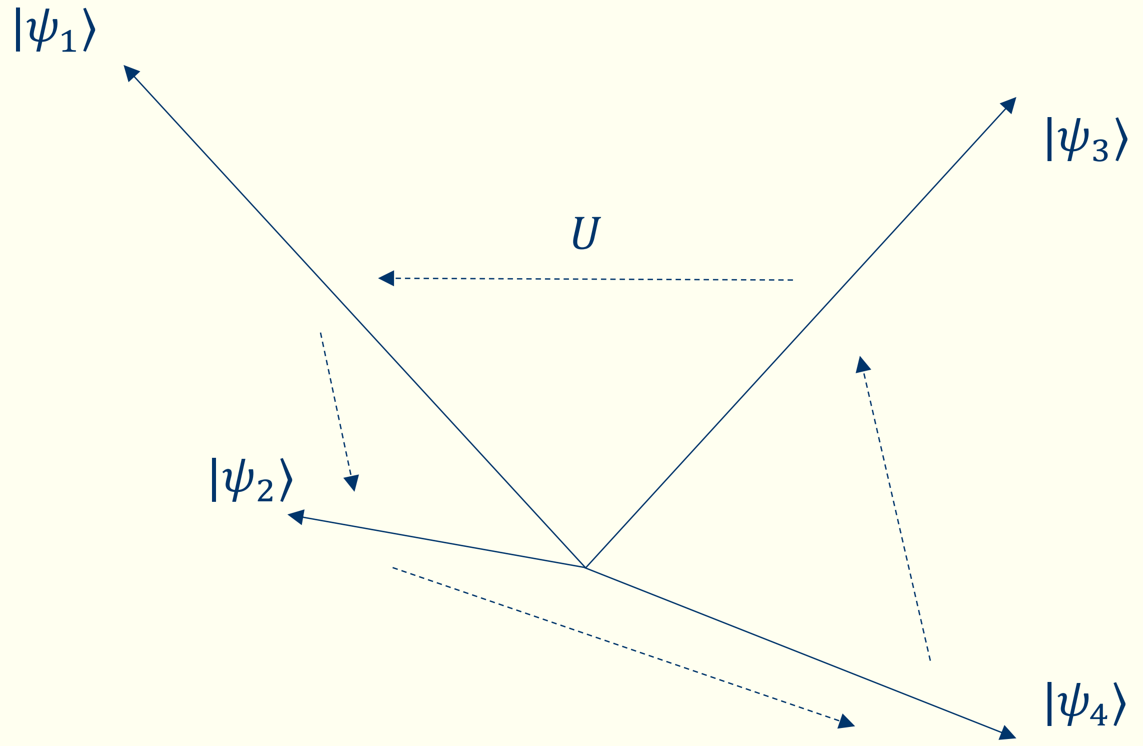
Swapping-distinguishing duality

Theorem [AAS'20]: You can efficiently implement swap between $|\phi\rangle$ and $|\psi\rangle$ if and only if you can efficiently distinguish between $|\phi\rangle + |\psi\rangle$ and $|\phi\rangle - |\psi\rangle$.



Generalized duality

Now say that I have many states $\{|\psi_x\rangle\}$ and a collection of mappings between then, $\{U_g\}$. Is there some measurement that characterizes the complexity of implementing all of those mappings?



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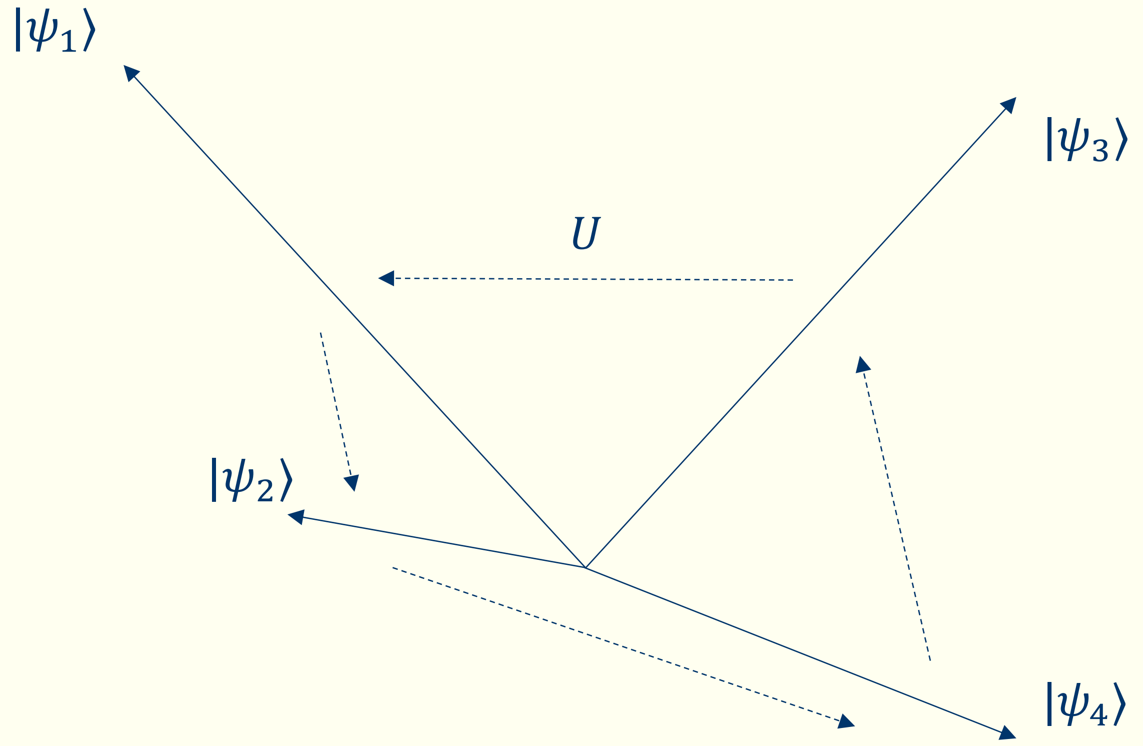
Representation of a group



Representations of groups

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What makes it a representation is that it also respects the group action:

$$\mathcal{R}(g)\mathcal{R}(h) = \mathcal{R}(gh)$$

Irreducible representations

For every group G, there is a dual group $\hat{G}$, and a collection of representations of G,

$$\{\rho^{\lambda}(g) : \lambda \in \hat{G}\}$$

Which we call the irreducible representations of G.

Irreducible representations

Important fact about irreps: For every representation $\mathcal{R}$ on vector space V, there is a decomposition of V into a direct sum of subspaces

$$V = \bigoplus_{\lambda,i} W_{\lambda,i}$$

Such that for every group element,

$$\mathcal{R}(g) \simeq \bigoplus_{\lambda,i} \varrho^{\lambda}(g)$$

Irreducible representations

Recall that if all of these unitaries commuted, we could simultaneously diagonalize all of them.

$$\mathcal{R}(g) = V^{\dagger}\left(\sum_{\lambda} \alpha_{\lambda}(g)|\psi_{\lambda}\rangle\langle\psi_{\lambda}|\right)V$$

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We call V the quantum Fourier transform

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Fourier extraction

If we write basis for each $W_{\lambda,i}$ as follows:

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The representation behaves identically on different copies of $W_{\lambda,i}$, making it difficult to figure out i in a black-box way.

Fourier extraction

Equivalent of a "coherent" measurement in the Fourier basis, up to the decomposition of different copies of the same W_λ.

$$\sum_j \alpha_j |\psi^\lambda_{i,j}\rangle \mapsto |\phi^\lambda_i\rangle|\lambda\rangle \otimes \sum_j \alpha_j |j\rangle$$

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Think of this as a hidden basis state that encodes information about i.

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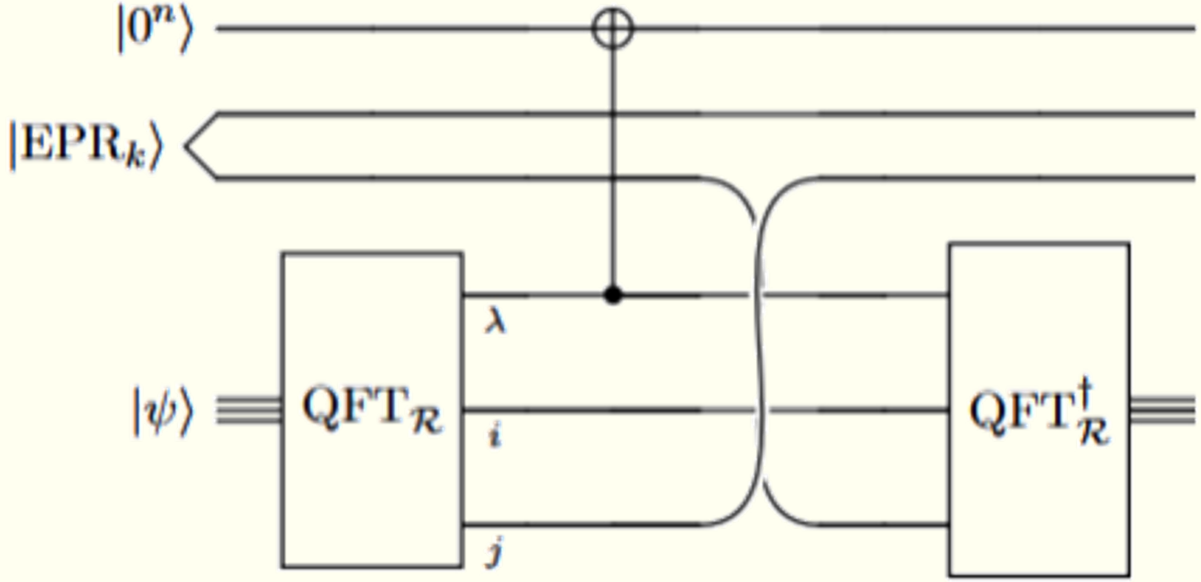
Here is what "ideal" Fourier extraction looks like:



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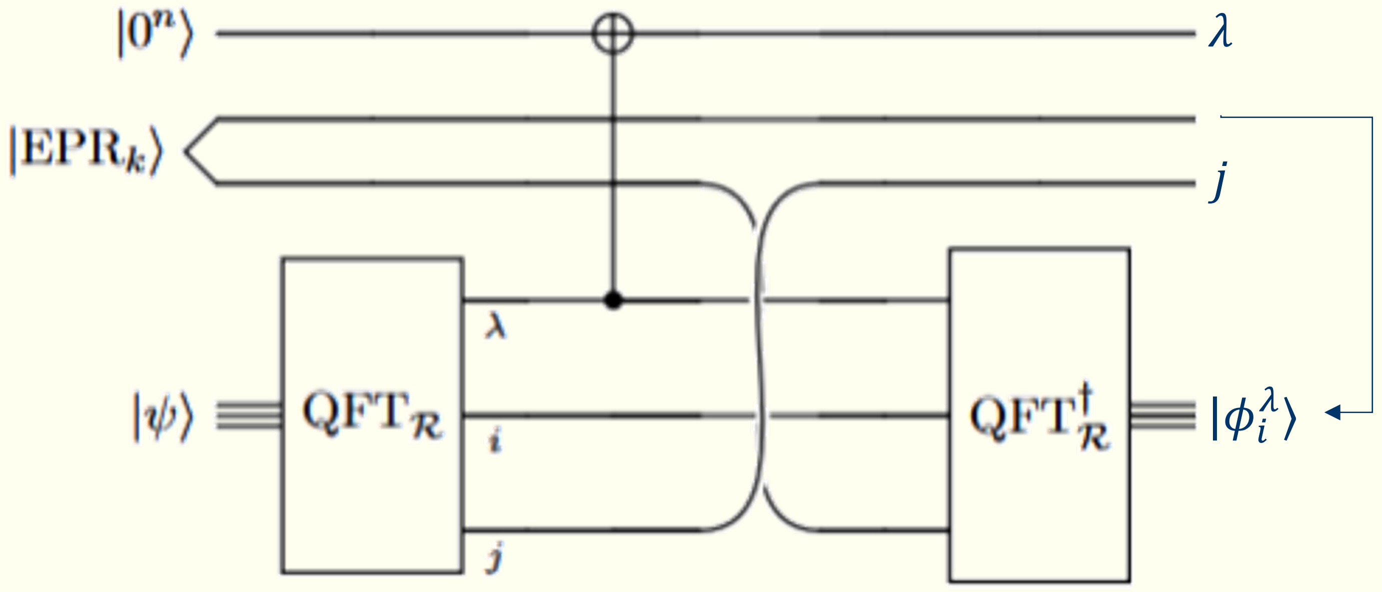

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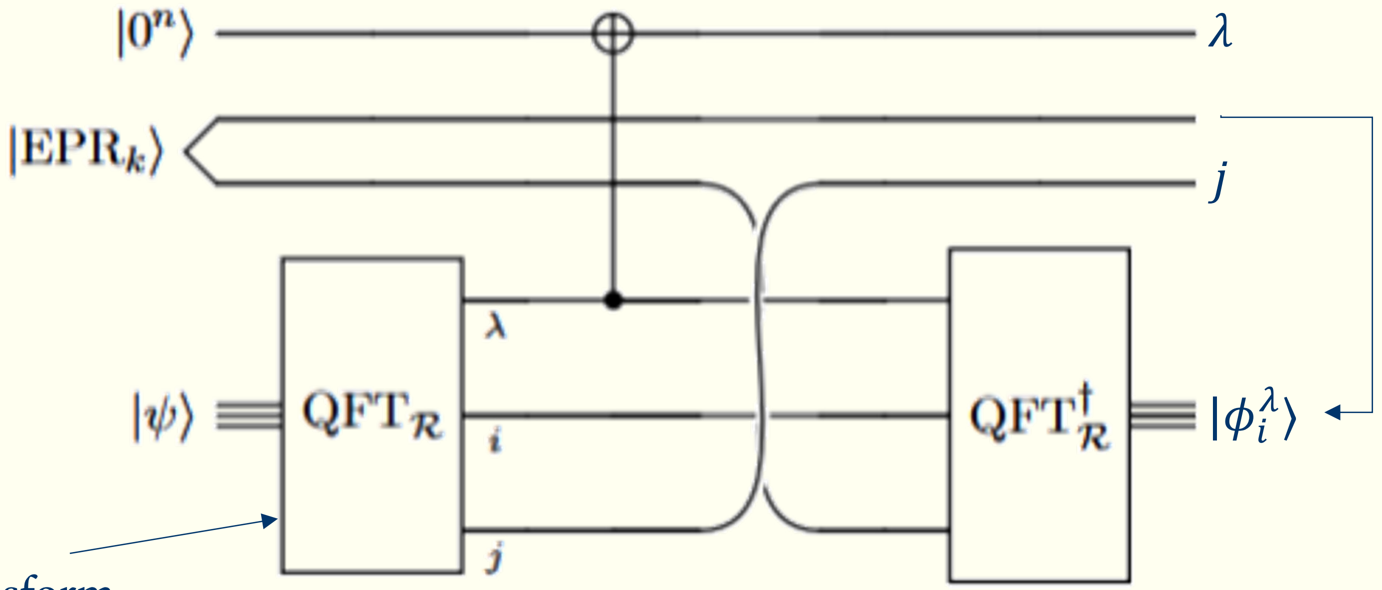
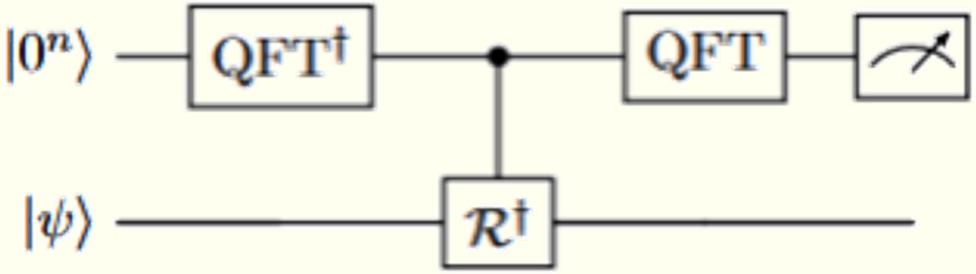


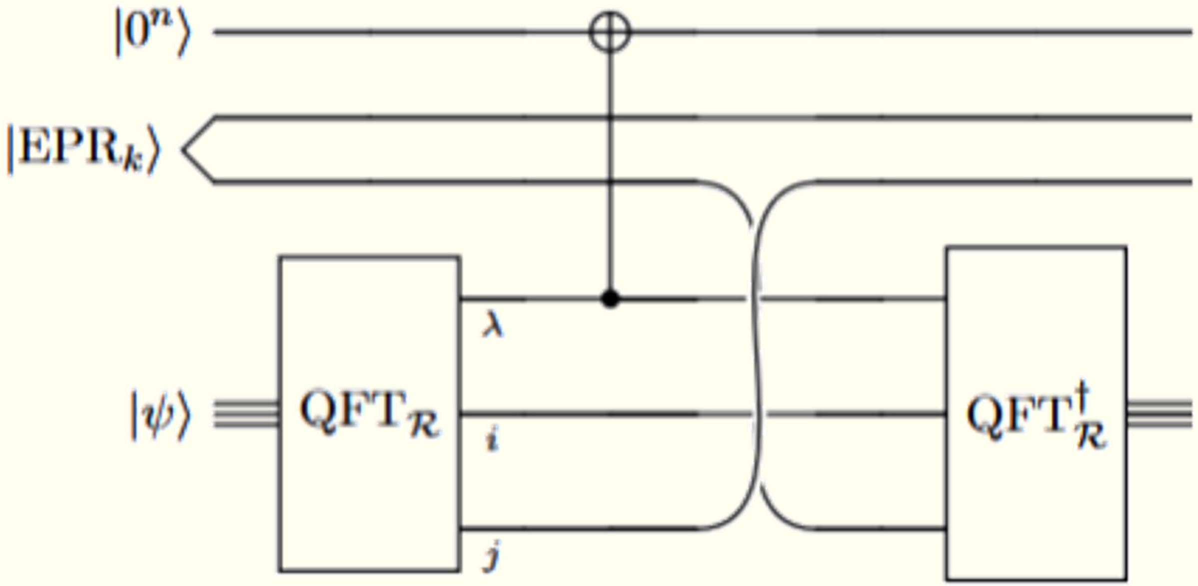
This is not the normal Fourier transform, but the Fourier transform for an arbitrary representation!

Fourier extraction

Turns out, it's equivalent to the following (where the measurement is only on the irrep label).



$\approx$



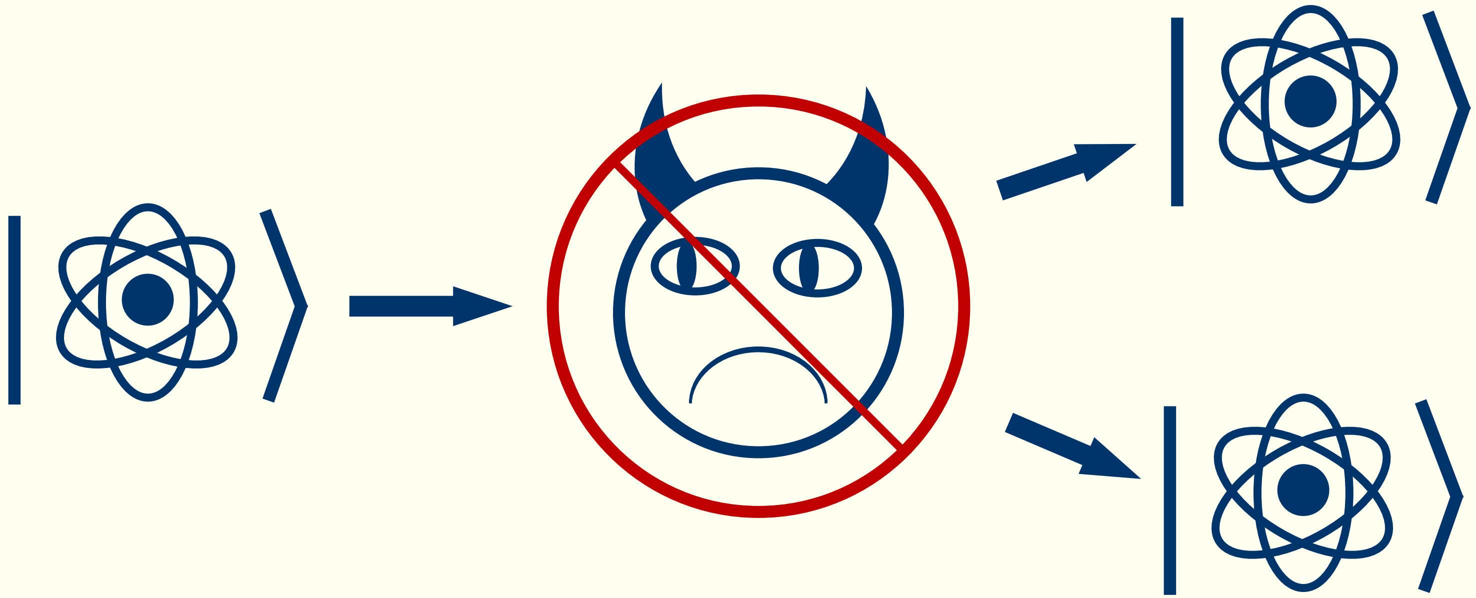
General duality theorem

Theorem: You can efficiently implement a group representation $\mathcal{R}$ if and only if you can efficiently implement Fourier extraction for the irreducible subspaces of $\mathcal{R}$.

Quantum Lightning from Non-Abelian Group Actions

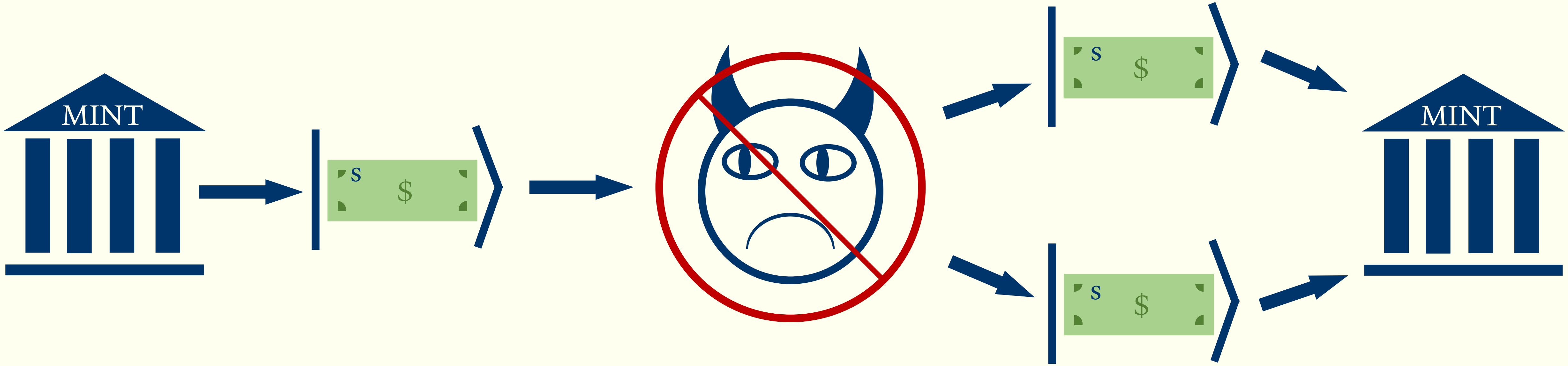
The no-cloning theorem

No cloning says no one can clone an **arbitrary** quantum state.



Private-key quantum money

Weisner (in 1970) used this idea to find states that could be efficiently minted, but could not be cloned by any adversary.



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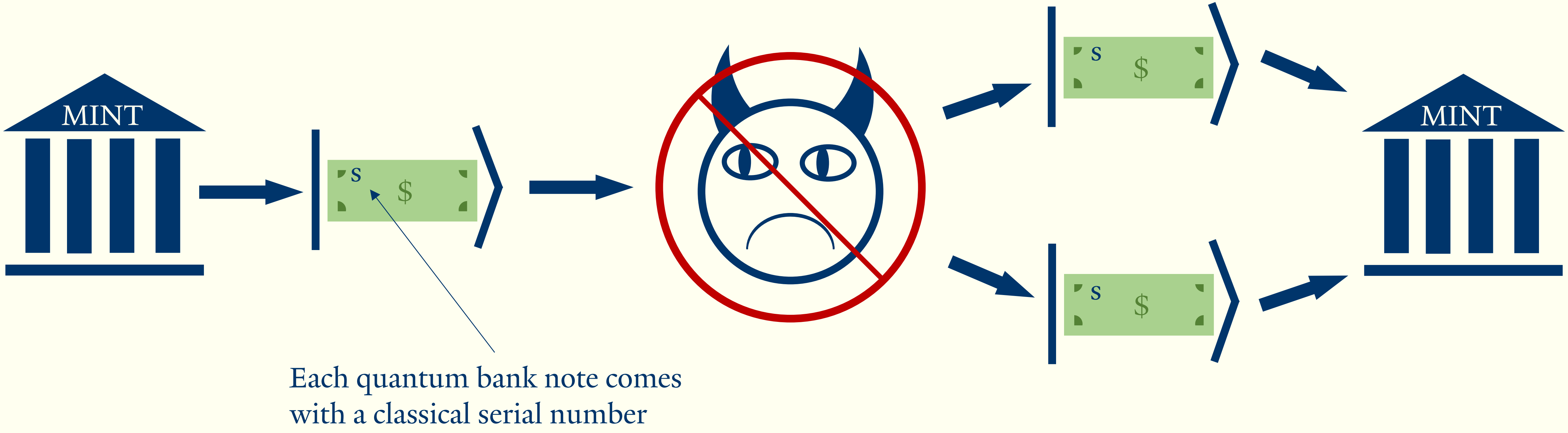

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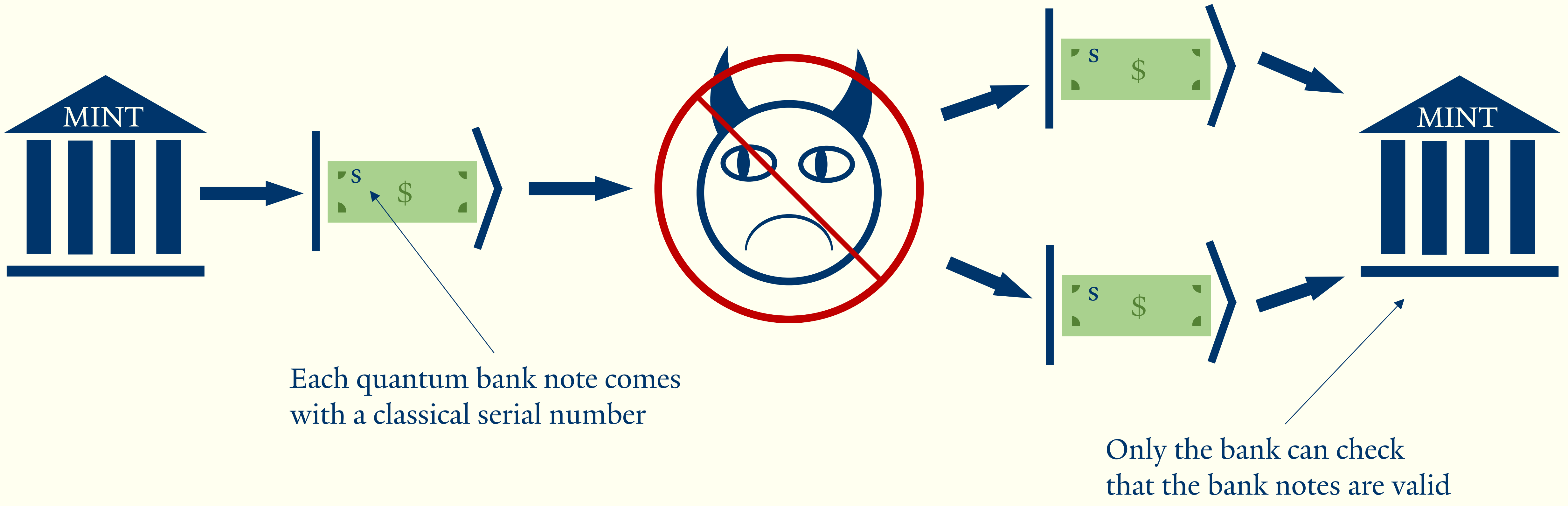
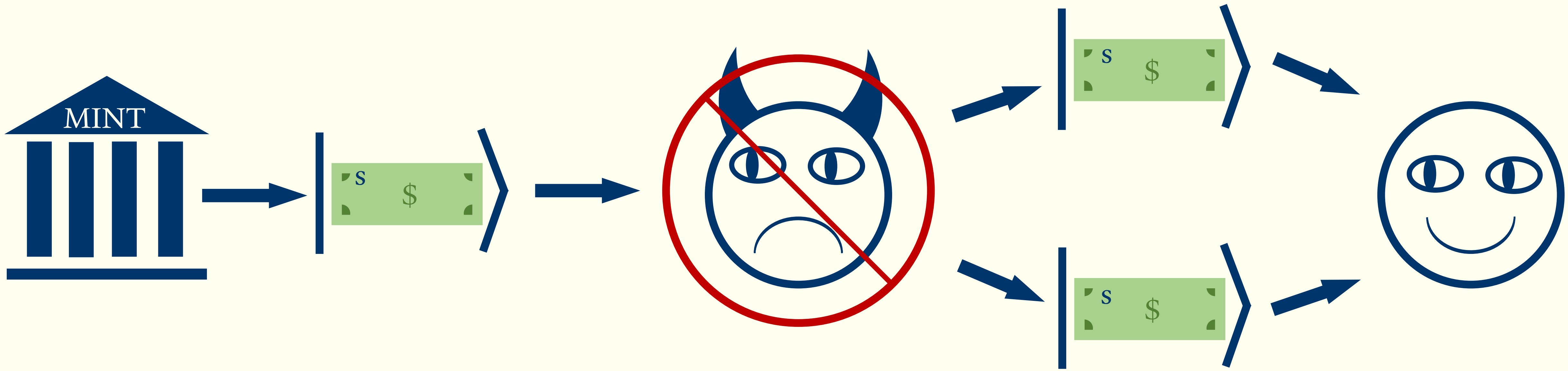


Public-key quantum money

Aaronson (2009) proposed quantum money that anyone can verify.



Public-key quantum lightning

Zhandry (2019) proposed a variant of quantum money that is "collision resistant".

Not even the mint can make two notes
that have the same serial number!

Adapted from Zhandry'23

Unfortunately, constructing quantum money has been really hard!

Only has conjectured security, or completely broken	Security in an idealized model	Security from a plain-model computational assumption

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Basically, the most power cryptography you could imagine, we don't know how to build this either

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Product in the group

Reminder: the quantum Fourier transform

Recall, we call any transformation that maps from the standard basis to the Fourier basis the "Fourier transform".

For the left-regular representation, $U_g|h\rangle \mapsto |gh\rangle$, one nice Fourier transform looks like this:

$$\mathrm{QFT} = \sum_{g\in G} \sum_{\lambda,i,j\in[\dim(W^\lambda)]} \sqrt{\frac{d_\lambda}{|G|}}\, \varrho^\lambda(g)_{i,j} |\lambda, i, j\rangle\langle g|$$

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Quantum lightning from group actions

In the construction, we'll need to start with a group action for a group that has an **efficient quantum Fourier transform**, e.g.

1. Any group whose size doesn't scale in n.
2. Dihedral group.
3. Symmetric group.

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Basically, measure in the Fourier basis, but only check the irrep label.

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Our answer: Distinguish between an operation that preserves your state (up to an arbitrary phase), and one that moves your state around.

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Preaction Indistinguishability:

It's hard to distinguish between a challenger that applies a random action, versus a challenger that applies a random action and a random preaction.

Security reduction (simplified)

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3. Do a SWAP test between the $|\phi_i^\lambda\rangle$ registers.

This test tells us if i stayed the same. A preaction will randomize $|\phi_i^\lambda\rangle$, but the (left) group action won't, so we can distinguish the two cases.

Quantum Lightning from Preaction Security

Theorem: Given any group action that is preaction secure, the scheme we described is a secure quantum lightning scheme.

Instantiations

Group action from the McEliece cryptosystem. The group action is the symmetric group, set elements are $n \times m$ matrices with entries from some finite field $\mathbb{F}$ (think: codewords of an error correcting code).

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We conjecture that this is preaction secure.

Open questions

- Can you reduce preaction security to a “standard” assumption, like discrete log being hard, or the hidden subgroup problem being hard?

- Can you build other things from preaction secure group actions? For example, one-shot signatures, or copy-protected software?

- Can we find an efficiently falsifiable variant of preaction indistinguishability? For example, if the group action had a trapdoor that allowed the challenger to implement a random preaction.