Learning the closest product state

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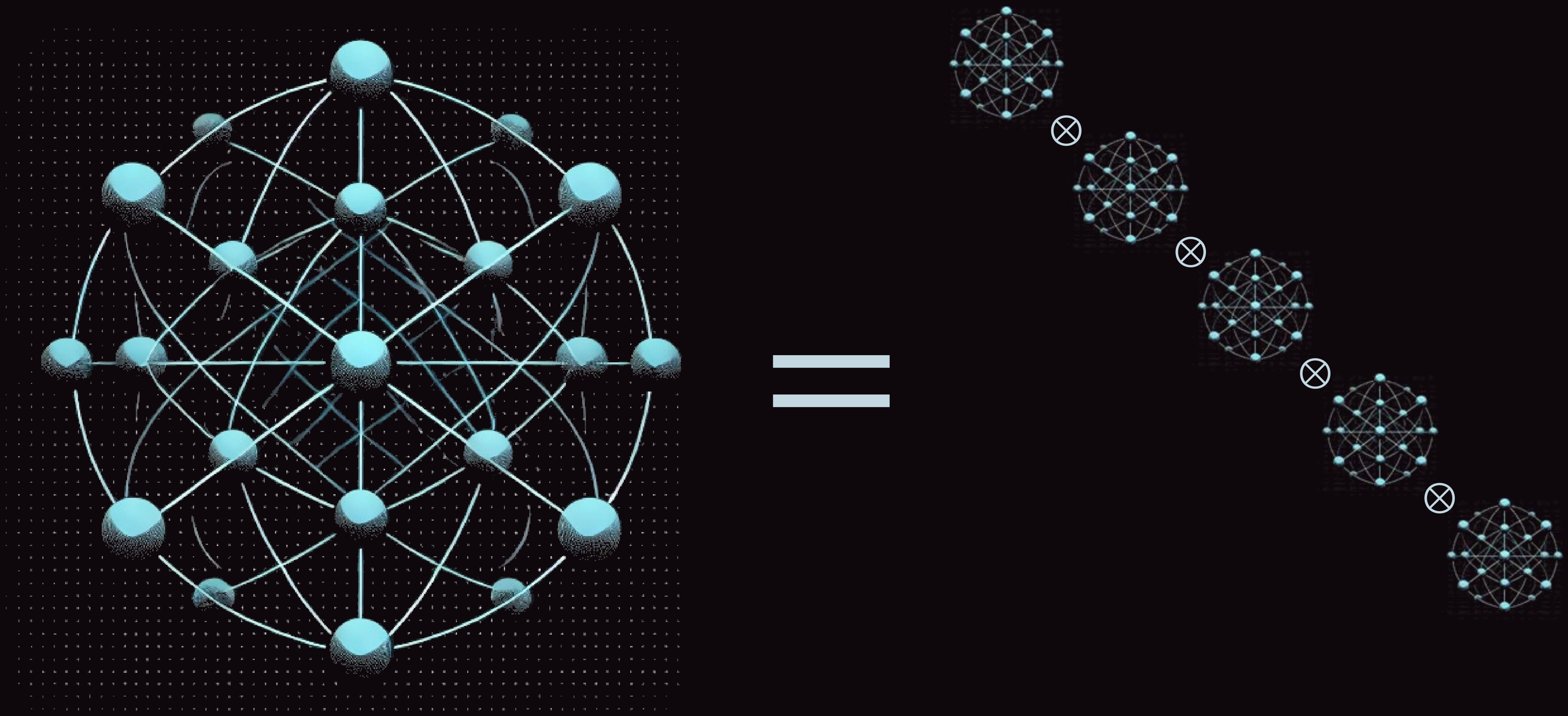
Quantum state learning

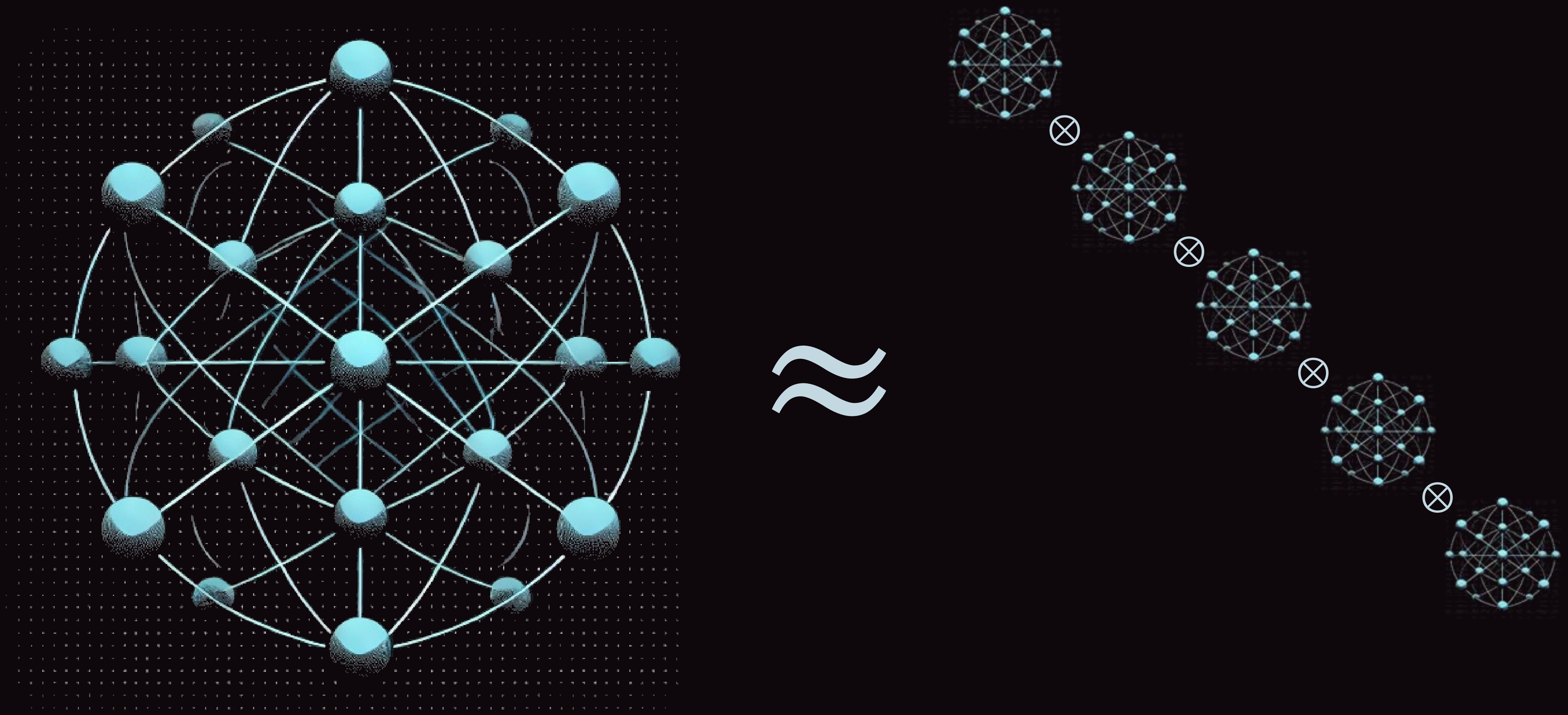
Typical set up:

Photo credits: OpenAI

Quantum state learning



Quantum state learning



Product state learning

Task: Learn a classical description of a product state.

Algorithm: Do state tomography on every register and output the tensor product the reduced states.

Product state learning

Consider the following input:

$$\sqrt{1-\epsilon}|0^n\rangle + \sqrt{\epsilon}|+^n\rangle$$

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Close to all 0's (product state), but every marginal isn't quite $|0\rangle$.

Product state learning

Moral: Even a little bit of misclassification error can change the nature of quantum state learning problems.

Agnostic quantum state learning

Given a model class C and copies of an arbitrary quantum state, output the description of the closest state in C to the state.

$$(C, \rho^{\otimes n}) \to \underset{|\psi\rangle \in C}{\operatorname{argmax}} \langle\psi|\rho|\psi\rangle$$

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If you don't care about runtime, shadow tomography solves this in $O(n \cdot \log^2(|C|) \cdot \epsilon^{-4})$ samples.

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Surprisingly, few computationally efficient algorithms exist, even for simple families like product states!

Learning the closest product state

Main result: We provide an algorithm for agnostic learning of product states that has sample complexity and time complexity that is

$$\mathrm{poly}\left(n^{poly\left(\frac{1}{\epsilon}\right)}\right)$$

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2. There are at most 2 orthogonal product states that have fidelity larger than ½ with ρ, and all of its reduced states.

Learning the closest product state

These observations motivate the following main loop:

For k from 1 through n:

Given a net $\{\pi_i\}$ for first k registers $(\langle \pi_i | \rho_{[k]} | \pi_i \rangle \geq \frac{1}{2}$ and $\langle \pi_i | \pi_j \rangle \approx 0)$,

Find a net for k+1 registers.

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High level algorithm:

1. Search in a small ball around $|\pi_i\rangle \otimes |\phi_{k+1}\rangle$ (the root candidate).

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The remaining technical challenge will be turning our objective into a low-degree polynomial and then optimizing that polynomial.

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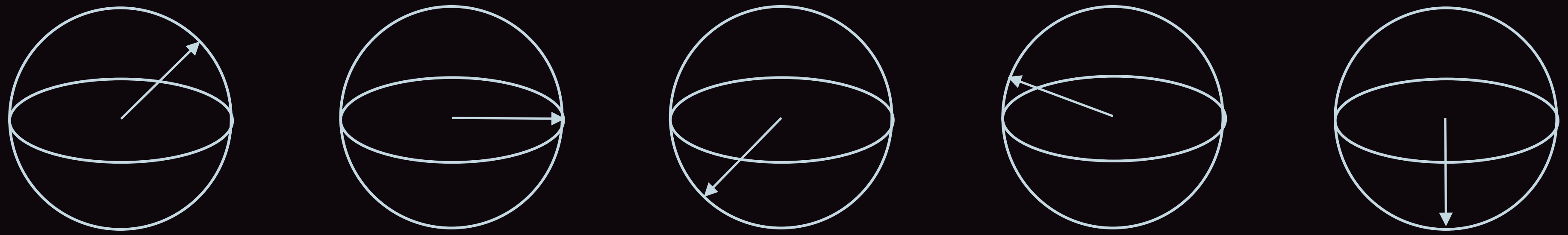

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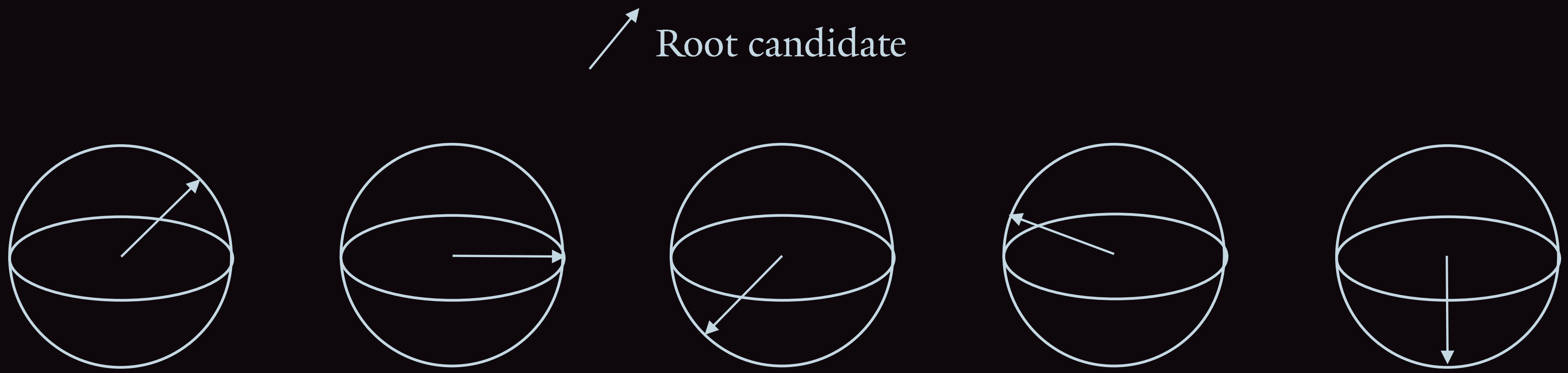
First rotate so that our root candidate is all 0's.



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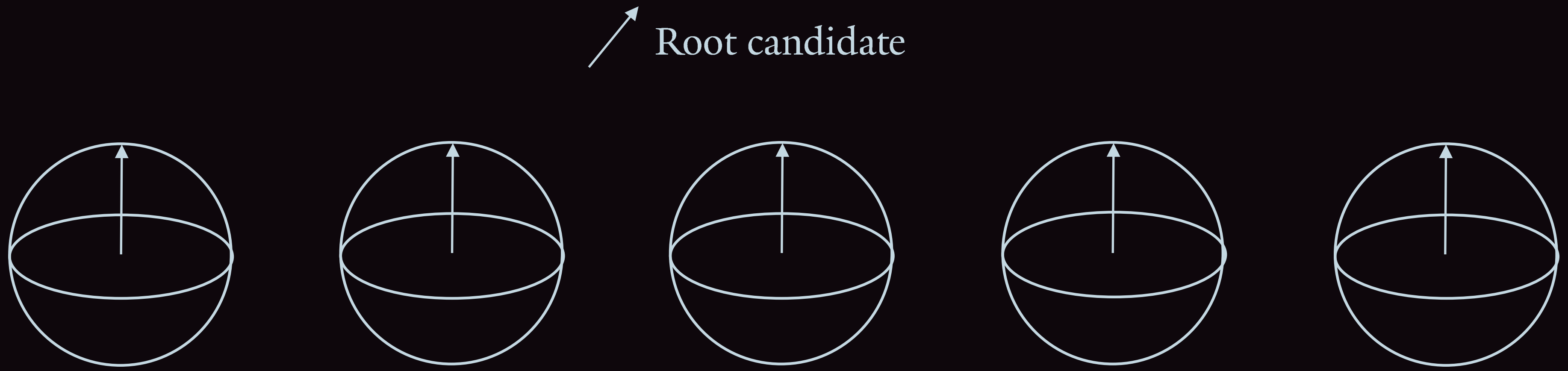

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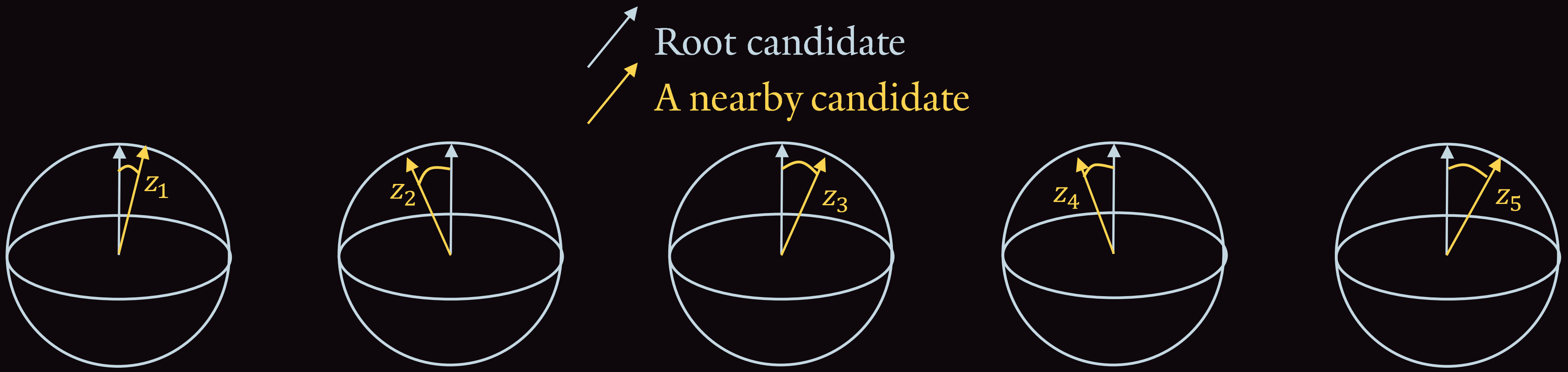

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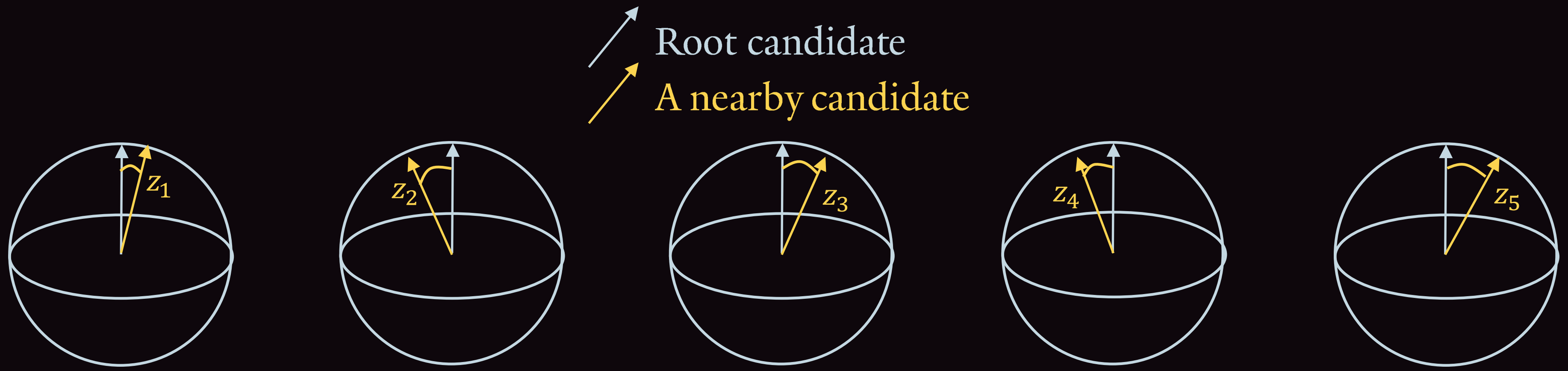

The "quantum part" of the algorithm will be to do tomography on the low-weight restriction of the input ρ.

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Maximizing fidelity ≈ maximizing the following polynomial

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Low Degree!

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3. Polynomial bond-dimension MPS

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Why? Turns out the connection to tensor optimization goes both ways.