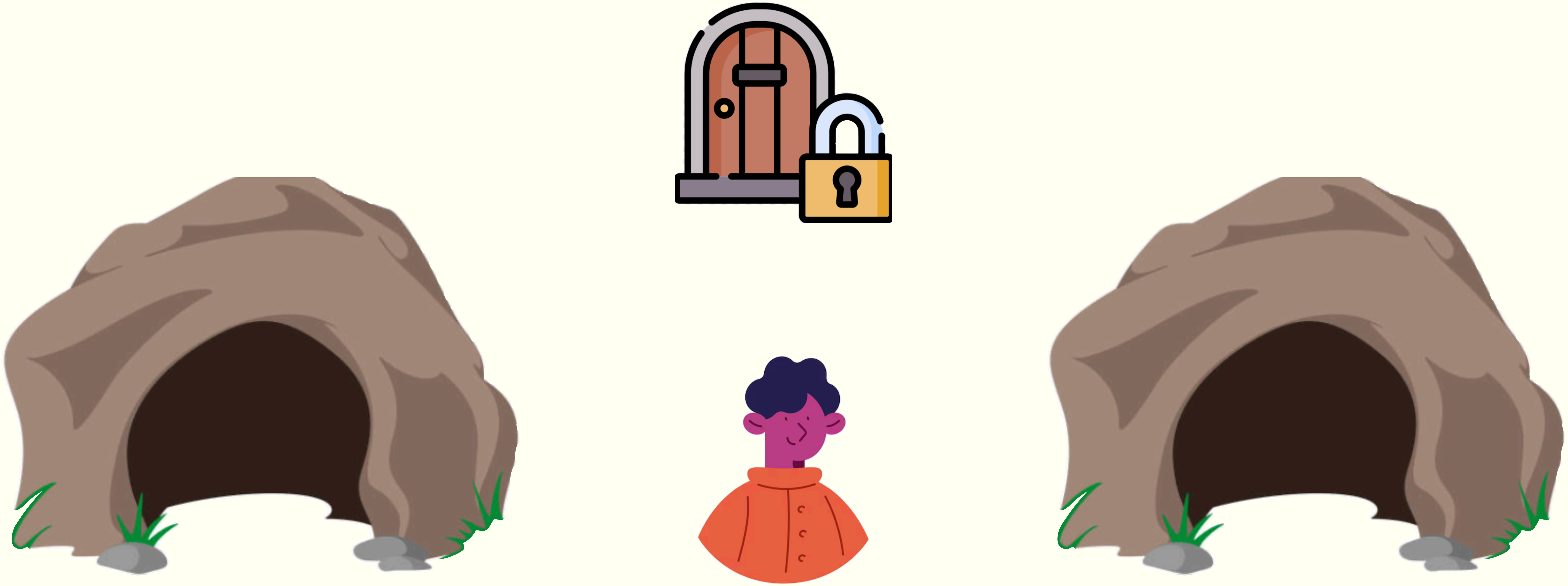
Local transformations of bipartite entanglement are rigid

John Bostanci, Tony Metger, and Henry Yuen

The cave game



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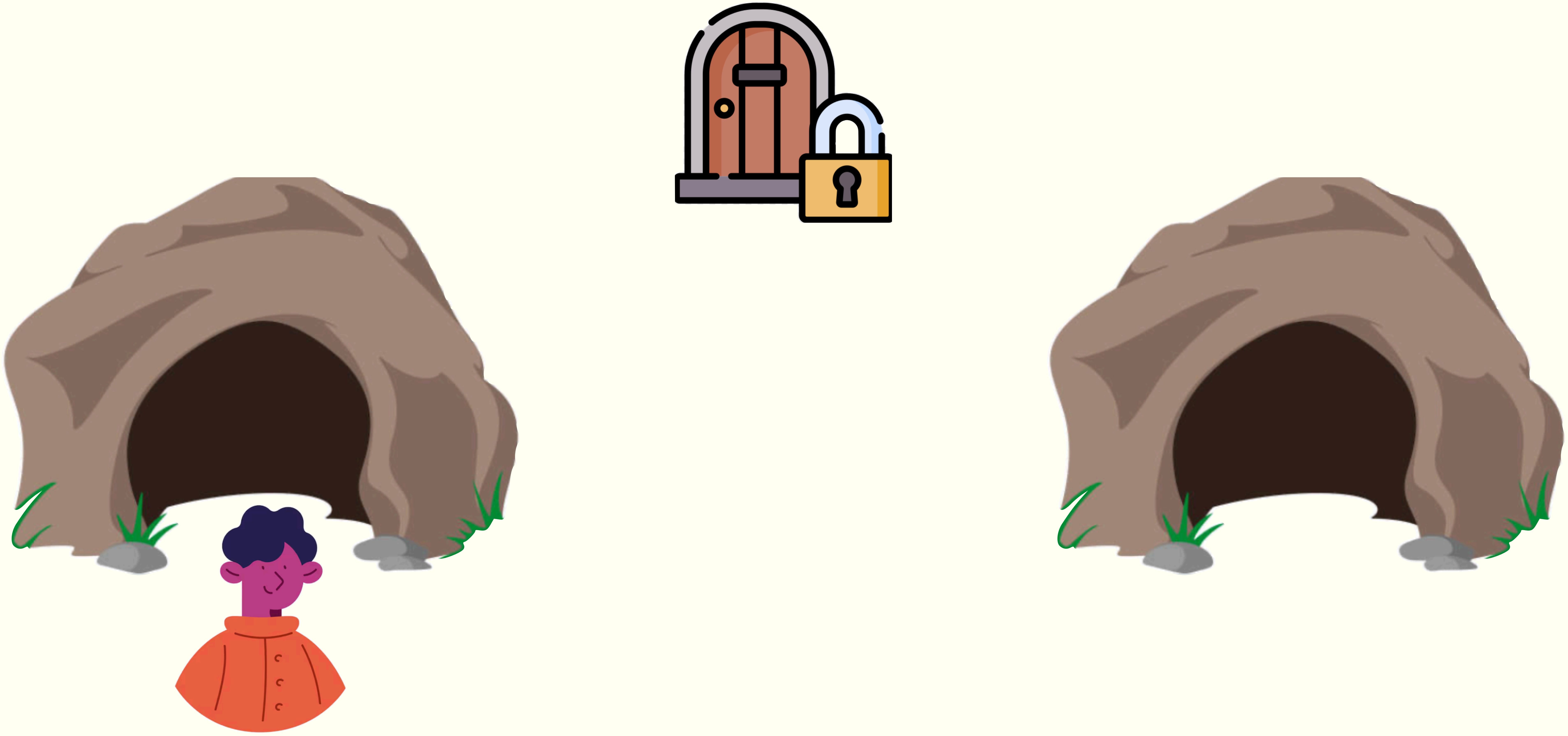

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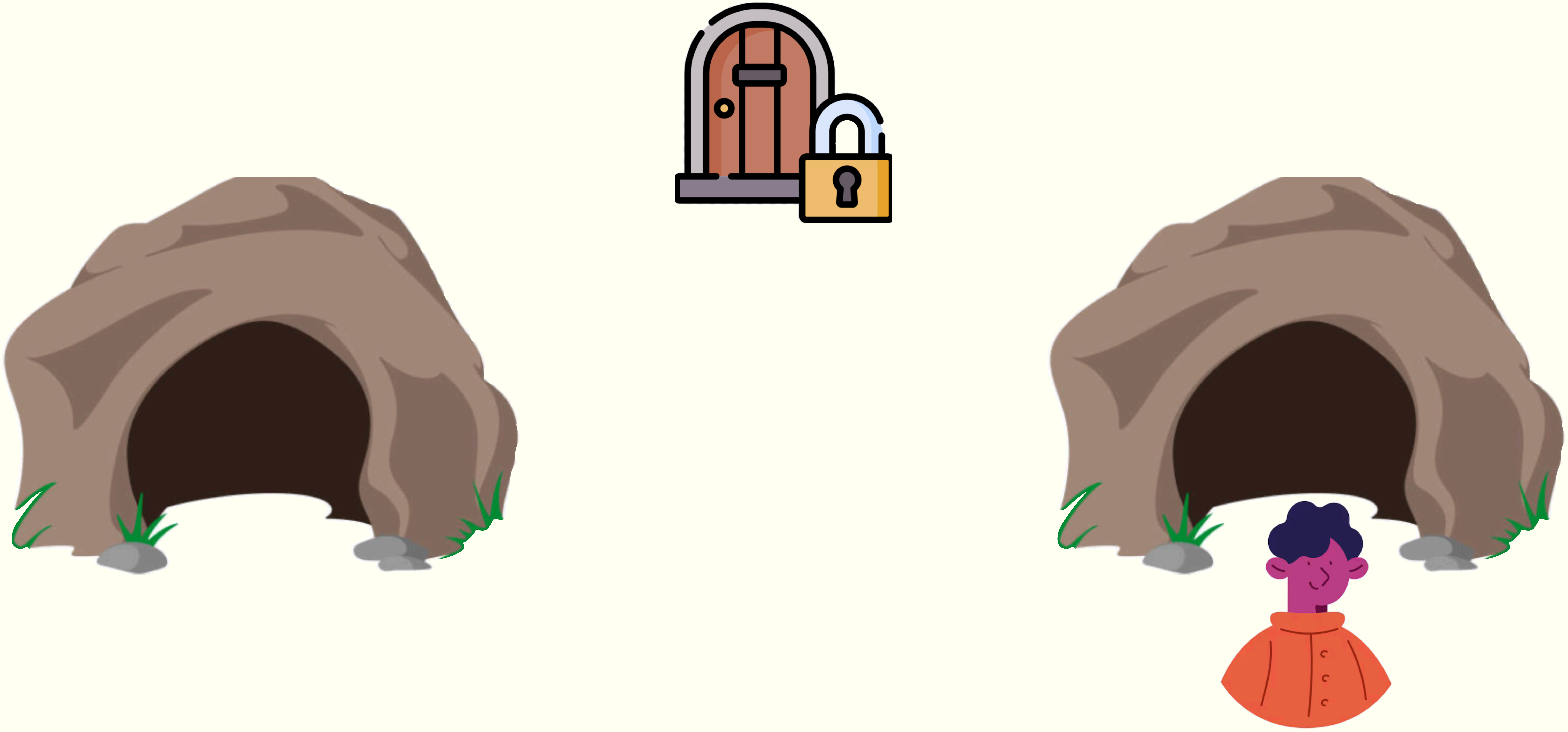

The cave game

After 100 tries, you might be convinced that your friend couldn't be doing this just by guessing what you're going to shout.

50 repetitions later

The cave game

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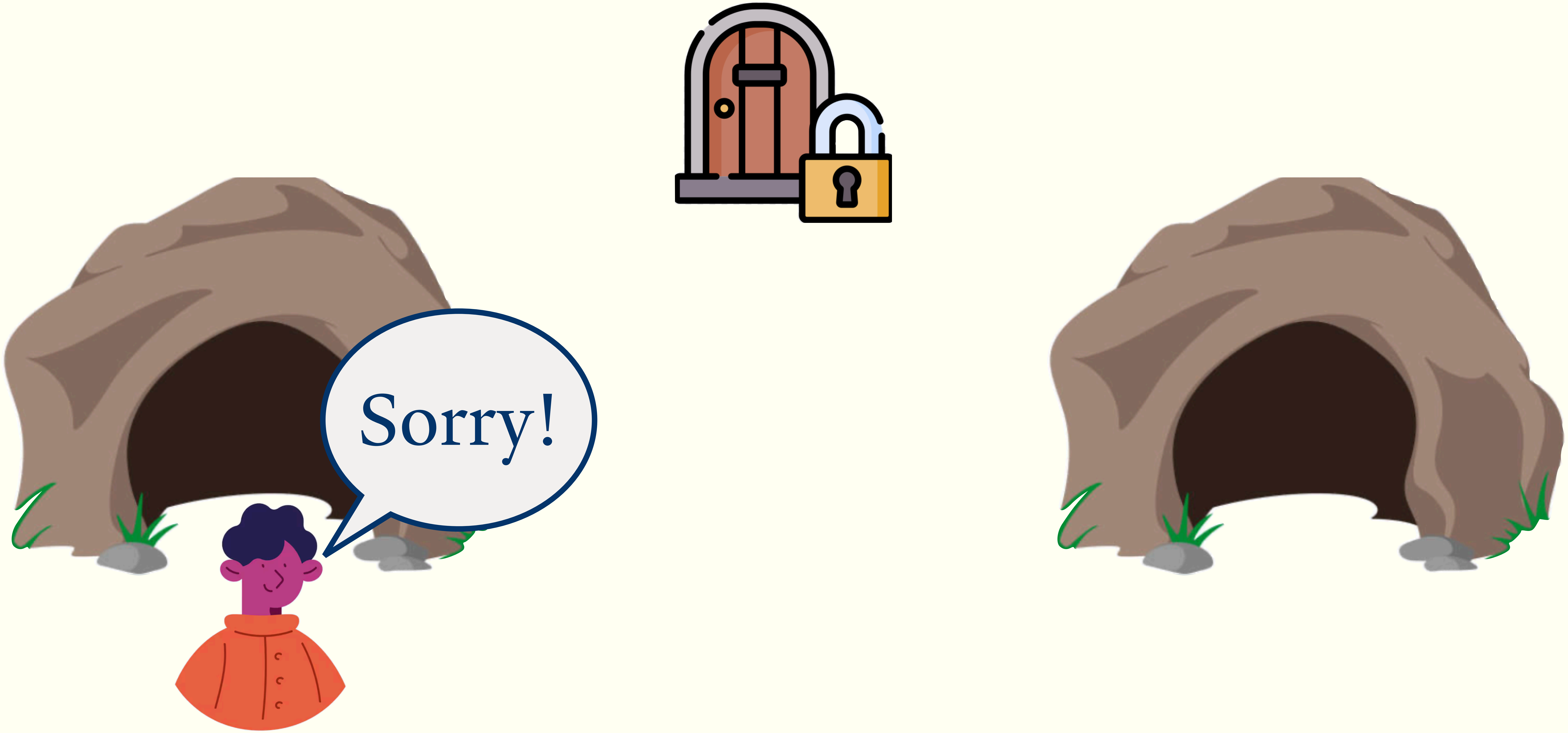


The cave game

How likely is it that your friend has the key?

The cave game

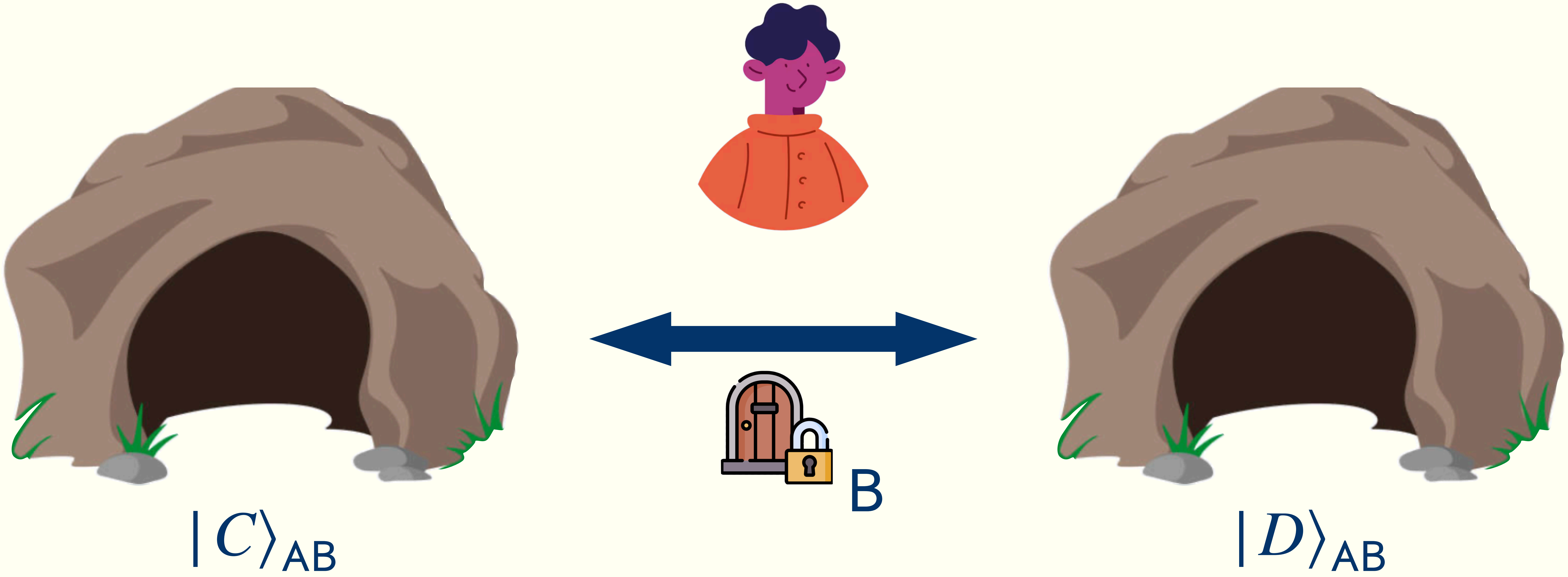
How likely is it that your friend has the key?

How would different doors/caves affect this?

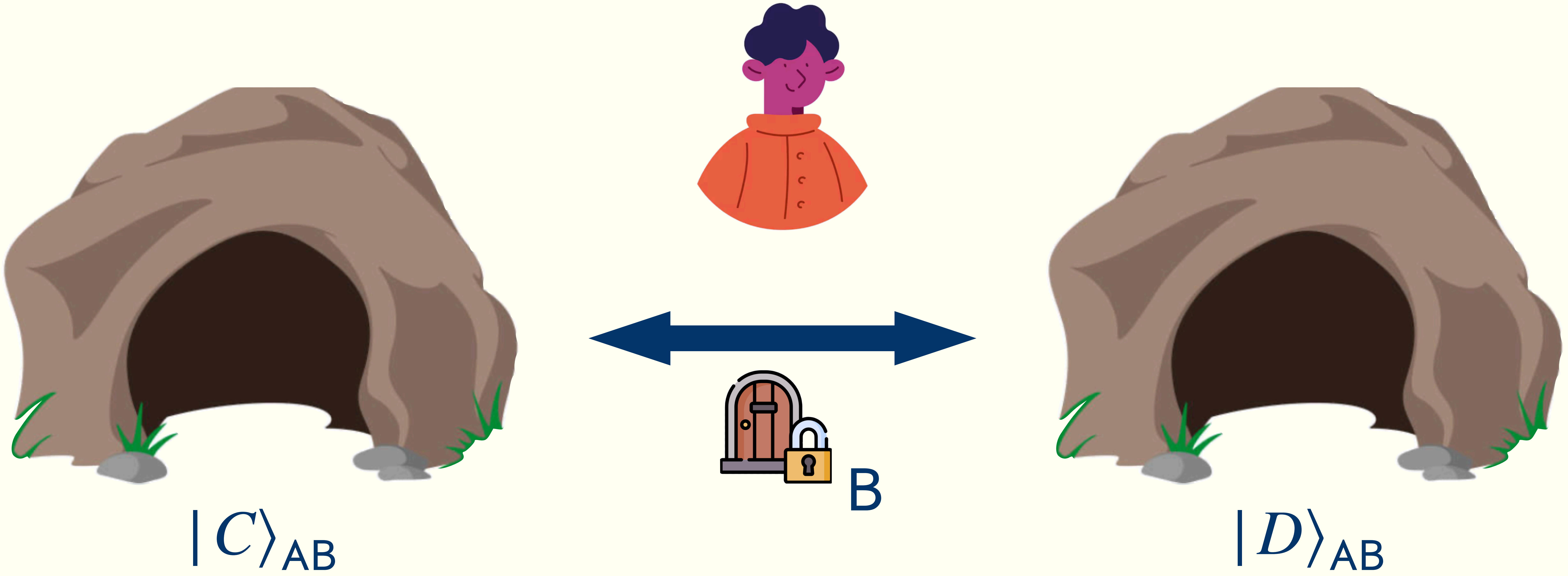
The quantum cave game

$|C\rangle_{\mathrm{AB}}$ $|D\rangle_{\mathrm{AB}}$

The **quantum** cave game

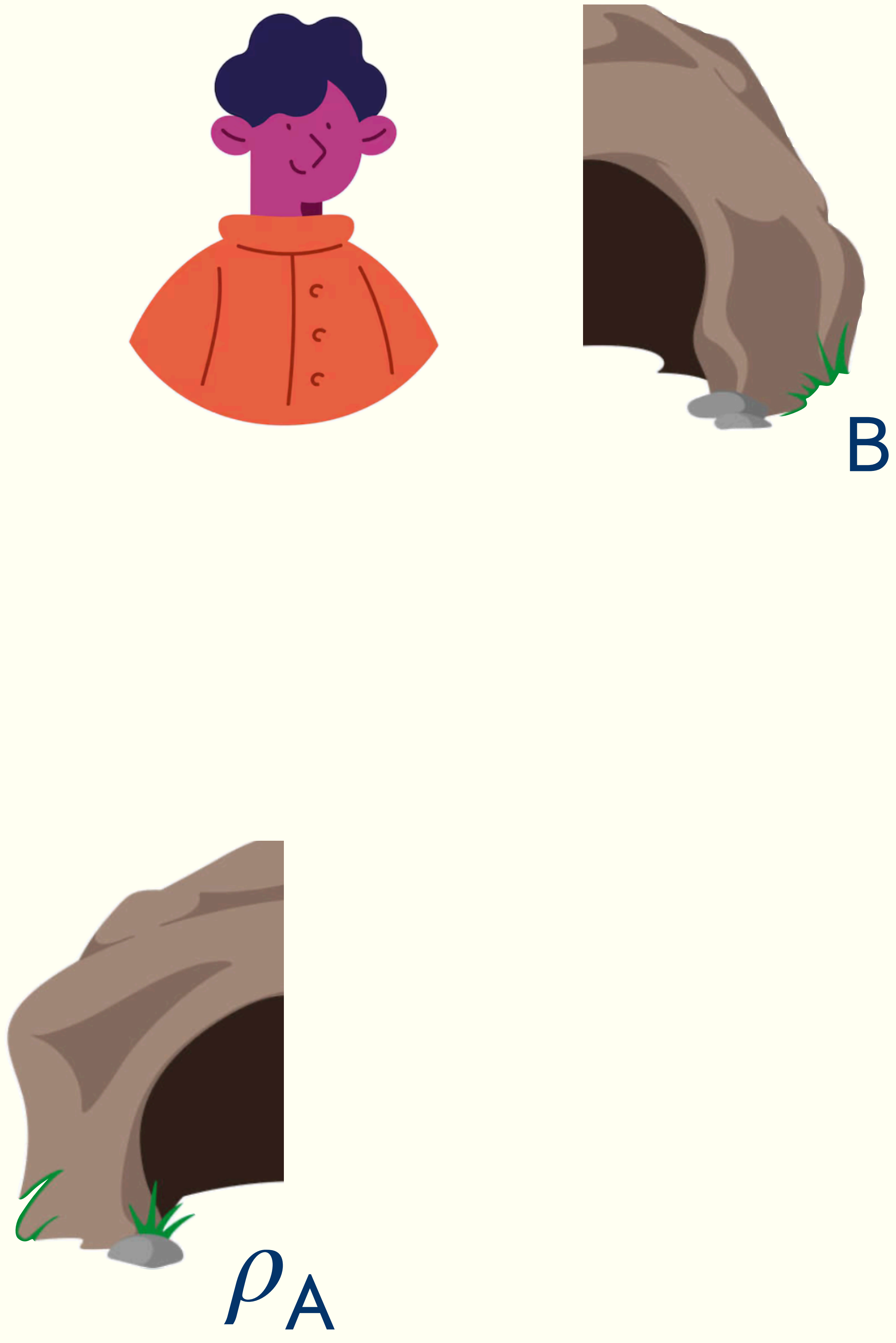


The quantum cave game



We call the best possible strategy for your friend the "Uhlmann transformation".

The quantum cave game



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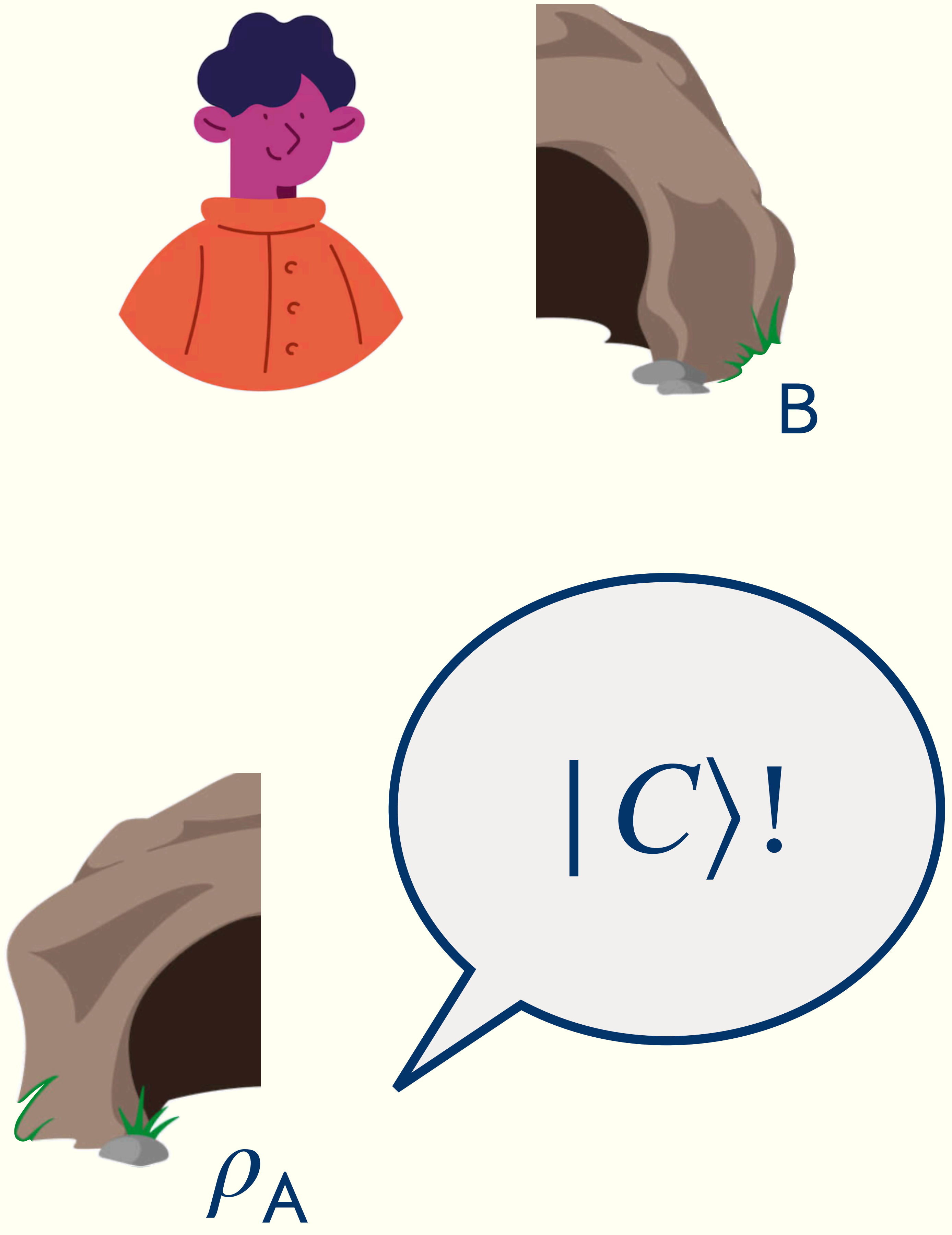

The quantum cave game

The quantum cave game rigidity

The maximum probability that your friend wins is:

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There is a canonical Uhlmann isometry, given by:

$$W = \mathrm{sgn}\left(\mathrm{Tr}_{\mathsf{A}}\left(|D\rangle\langle C|\right)\right)$$

The quantum cave game rigidity

Question 1: Does winning the quantum cave game with probability

$$\frac{1}{2} + \frac{1}{2} F^2(\rho, \sigma) - \epsilon$$

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Question 2: What are the properties of $|C\rangle$ and $|D\rangle$ that affect how confident you are about your friends ability to implement an Uhlmann transformation?

The quantum cave game rigidity

Theorem: Let $|C\rangle_{\mathsf{AB}}, |D\rangle_{\mathsf{AB}}$ be two quantum states with reduced states on A, ρ, σ. Then for all unitaries U such that

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Here, $\delta(\epsilon) = \left(\frac{2\kappa}{\eta}\right) \cdot \epsilon$, where κ, η are:

$$\kappa = \left\| \rho^{-1/2} \cdot \mathrm{Image}(\rho^{1/2}\sigma\rho^{1/2}) \cdot \rho^{1/2} \right\|^2_{\mathrm{op}} \quad \text{and} \quad \eta = \lambda_{\min}(\rho^{-1} \# \sigma)\,.$$

The quantum cave game rigidity, idea

Proof idea:

1. For a fixed error ϵ, we can imagine searching for the furthest possible unitary from the canonical Uhlmann transformation that achieves $F^2(\rho, \sigma) - \epsilon$ using a SDP.

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3. When we find such a solution and plug it into the dual, we get the bound we want.

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- κ (Obliqueness): Kind of accounts for how different the left versus right cave entrances (i.e., going from left to right could be very different than going from right to left).

- η (Spectral gap): Accounts for the fact that $|C\rangle$ and $|D\rangle$ might have parts where they barely overlap. Ignoring those parts will not change the success probability of the quantum cave game much, but could change the unitary a lot.

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Theorem: For every $\kappa \geq 1$ and all ϵ, there is a pair of states with $\eta \geq 1$ and $\kappa = \left\| \rho^{-1/2} \cdot \mathrm{Image}(\rho^{1/2}\sigma\rho^{1/2}) \cdot \rho^{1/2} \right\|_{\mathrm{op}}^{2}$, and a unitary U such that,

$$\|\mathrm{id} \otimes (W - U)W^*W|C\rangle\|^2 \geq \kappa\epsilon^2$$

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Theorem: For every η, there is a pair of states whose matrix geometric mean has smallest eigenvalue η, and there is a transformation that saturates the bound, i.e.

$$\|\mathrm{id} \otimes (W - U)W^*W|C\rangle\|^2 \geq 2\epsilon/\eta$$

The quantum cave game rigidity, dependencies

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Theorem: For any states $|C\rangle$ and $|D\rangle$, and any η, you can find nearby states $|\tilde{C}\rangle$ and $|\tilde{D}\rangle$ such that the spectral gap is at least η and $|\langle D|\tilde{D}\rangle|^2 \geq 1-\eta^2$, $|\langle C|\tilde{C}\rangle|^2 \geq 1-\eta^2$.

Application: Representation theory

Our rigidity theorem is actually pretty general, and allows you to convert other notions of "operational closeness" to "structural closeness". Let's go through one example!

Application: Representation theory

Say that you have an approximate representation U_g, i.e., $\frac{1}{d}\mathbb{E}_{g,h}\left[\|U_g U_h - U_{gh}\|_1^2\right] \leq \epsilon$

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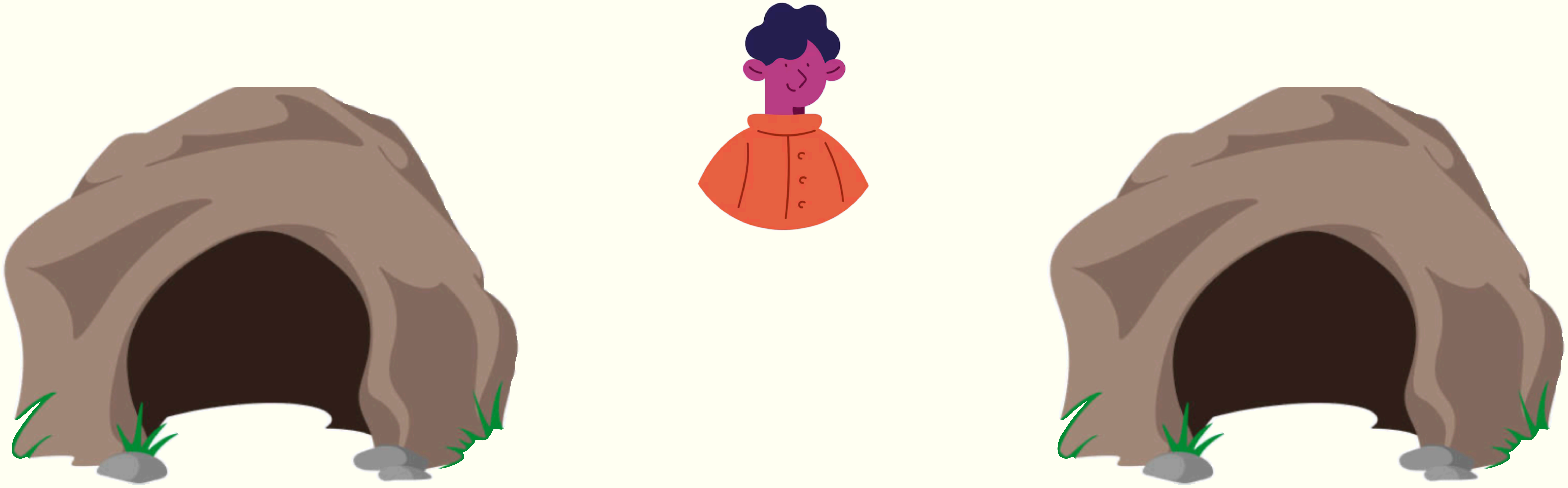

$$|C\rangle = \frac{1}{|G|}\sum_{g,h\in G}\left(\mathrm{id}\otimes U_g\right)|\mathrm{EPR}\rangle_{\mathsf{AB}_1}|g\rangle_{\mathsf{B}_2}|h\rangle_{\mathsf{B}_3}$$

$$|D\rangle = \frac{1}{|G|}\sum_{g,h\in G}\left(\mathrm{id}\otimes U_{hg}\right)|\mathrm{EPR}\rangle_{\mathsf{AB}_1}|g\rangle_{\mathsf{B}_2}|h\rangle_{\mathsf{B}_3}$$

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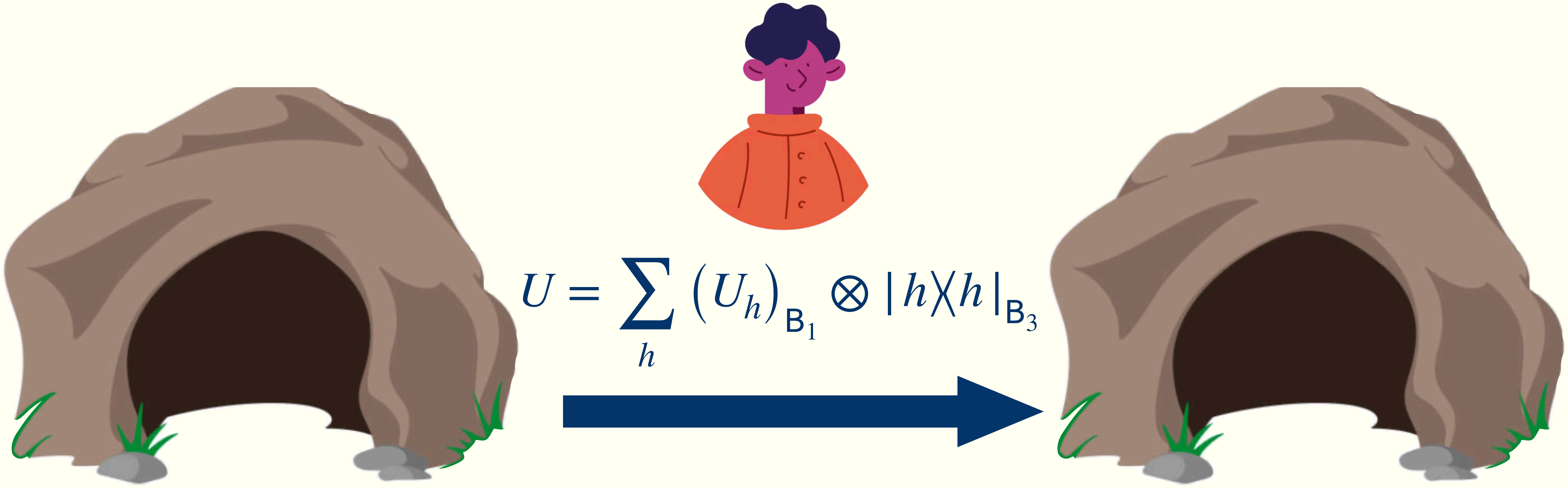

$$U = \sum_h (U_h)_{\mathsf{B}_1} \otimes |h\rangle\langle h|_{\mathsf{B}_3}$$

Corollary (Stability of approximate representations): There exists an isometry V and exact representation R such that $\frac{1}{d}\mathbb{E}_g\left[\|U_g - V^*R(g)V\|_1^2\right] \leq \epsilon$

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- We proved a rigidity theorem for Uhlmann transformations (quantum cave games).
- Our rigidity has dependence on two parameters, a “spectral gap”, and “obliqueness”, but these seem like they must appear in the rigidity theorem.
- Our rigidity theorem allows us to relate other notions of “operational closeness” to notions of “structural closeness”, like for approximate representations.

Next steps

1. Can we relate other notions of stability to Uhlmann transformations? For example, is CHSH rigidity a consequence of the rigidity of the Uhlmann transformation for some pair of states? What about general algebra's and non-local games?

2. There is a way to round states to nearby states so that η (the minimum eigenvalue of the matrix geometric mean) is well behaved, does the same exist for κ?

Thanks for listening!