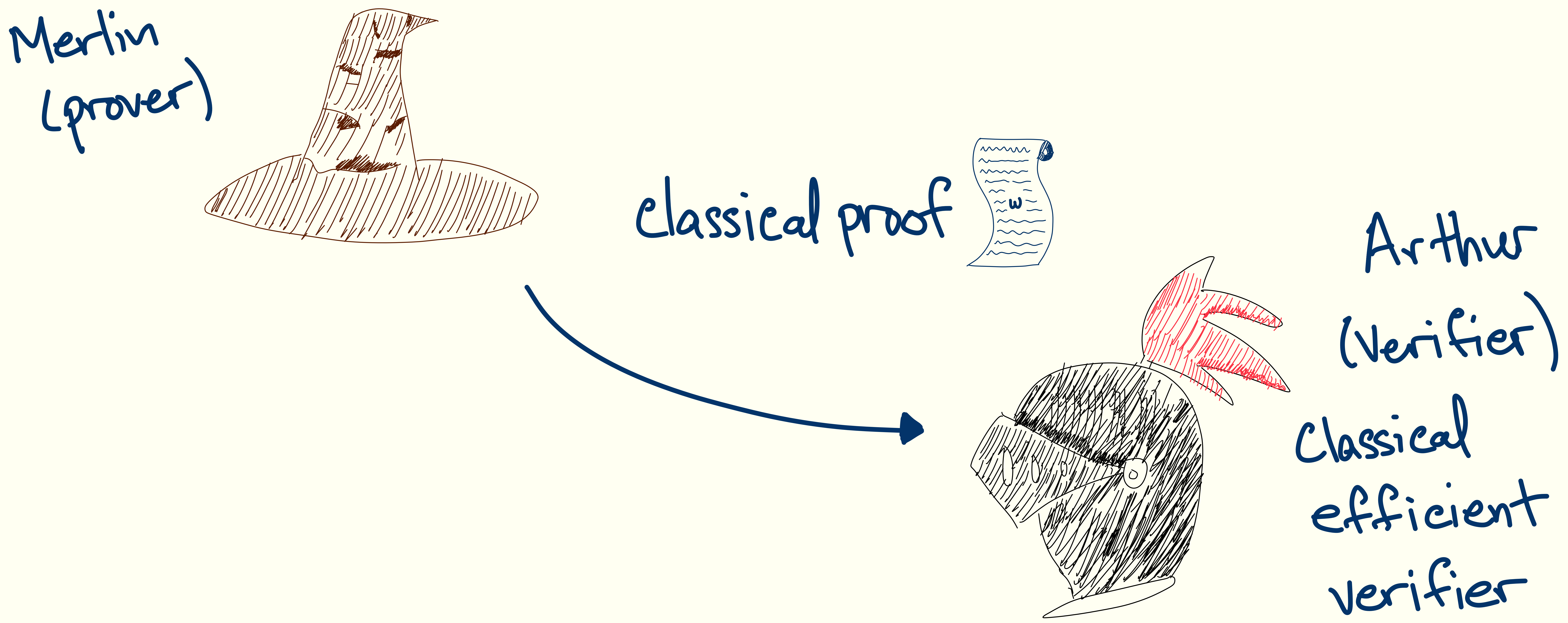
Separating QMA from QCMA with a classical oracle

John Bostanci, Jonas Haferkamp, Chinmay Nirkhe, and Mark Zhandry

How do model the power of proofs?

In complexity theory, the class NP captures the kinds of problems that we hope to be able to prove to one another.



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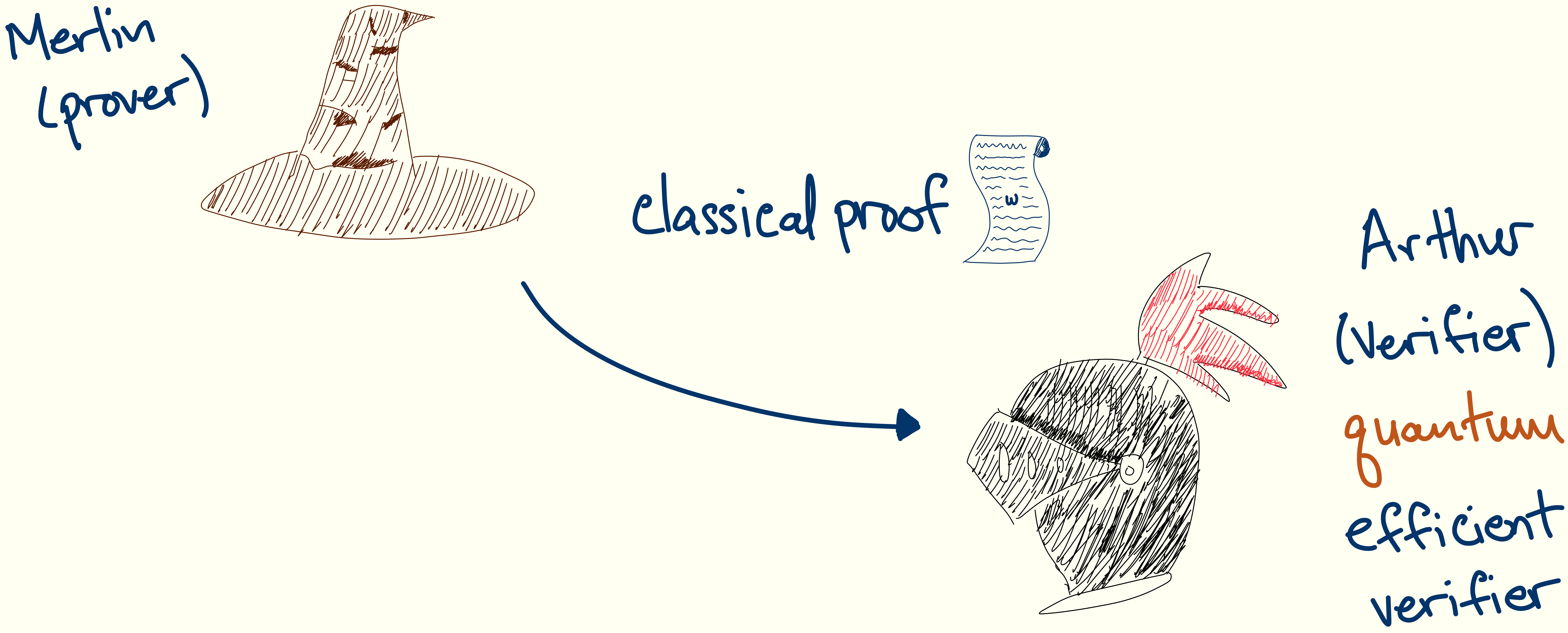
With quantum computers, we can compare the relative powers of quantum proofs and classical proofs. QCMA captures the kinds of problems we could prove classically.



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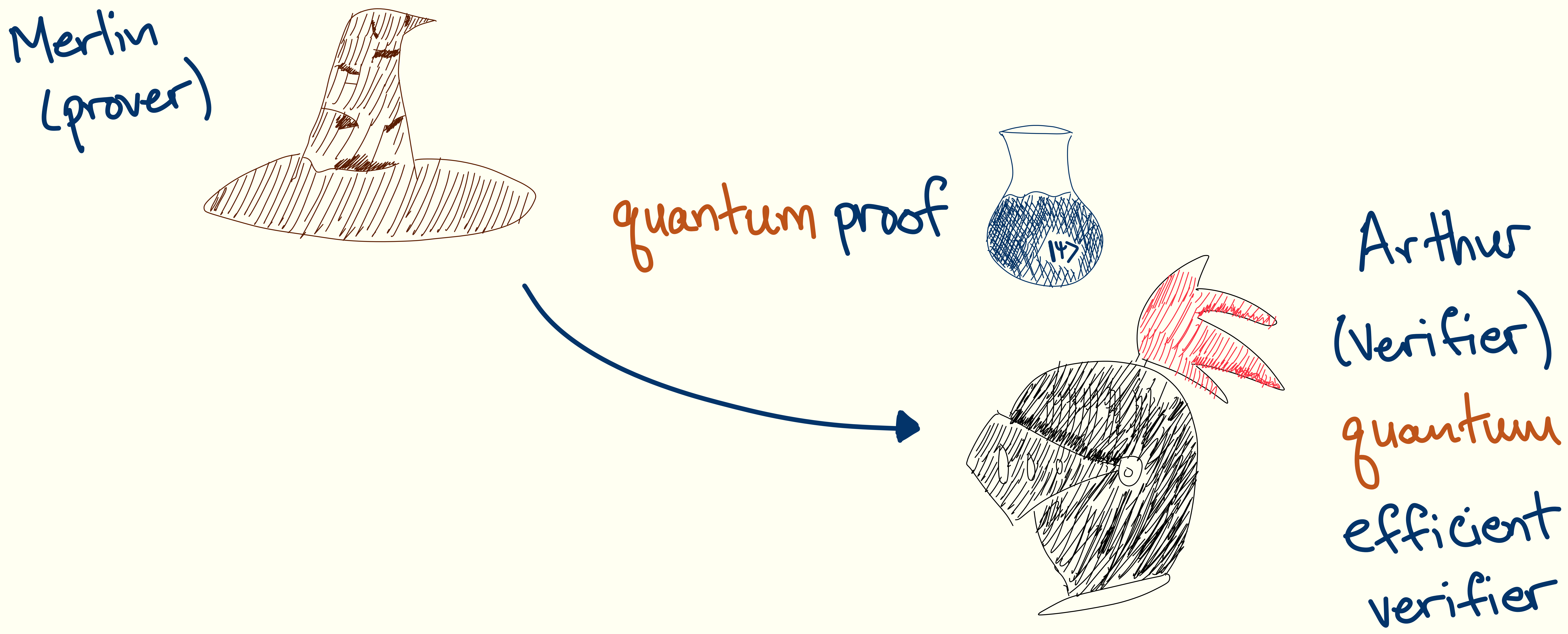

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- If QCMA = QMA, then anything you could verify about a ground state could be written down as a classical string!
- Otherwise, there must be something interesting about ground states you can only learn from having a copy of the state!

Why care about QMA versus QCMA?

Studying QMA versus QCMA is kind of like asking:

Are all "relevant" properties of quantum ground states possible to write down classically?

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  - Quantum oracle separation: Everyone gets access to a family of unitaries $\{U_n\}_{n=1}^{\infty}$.
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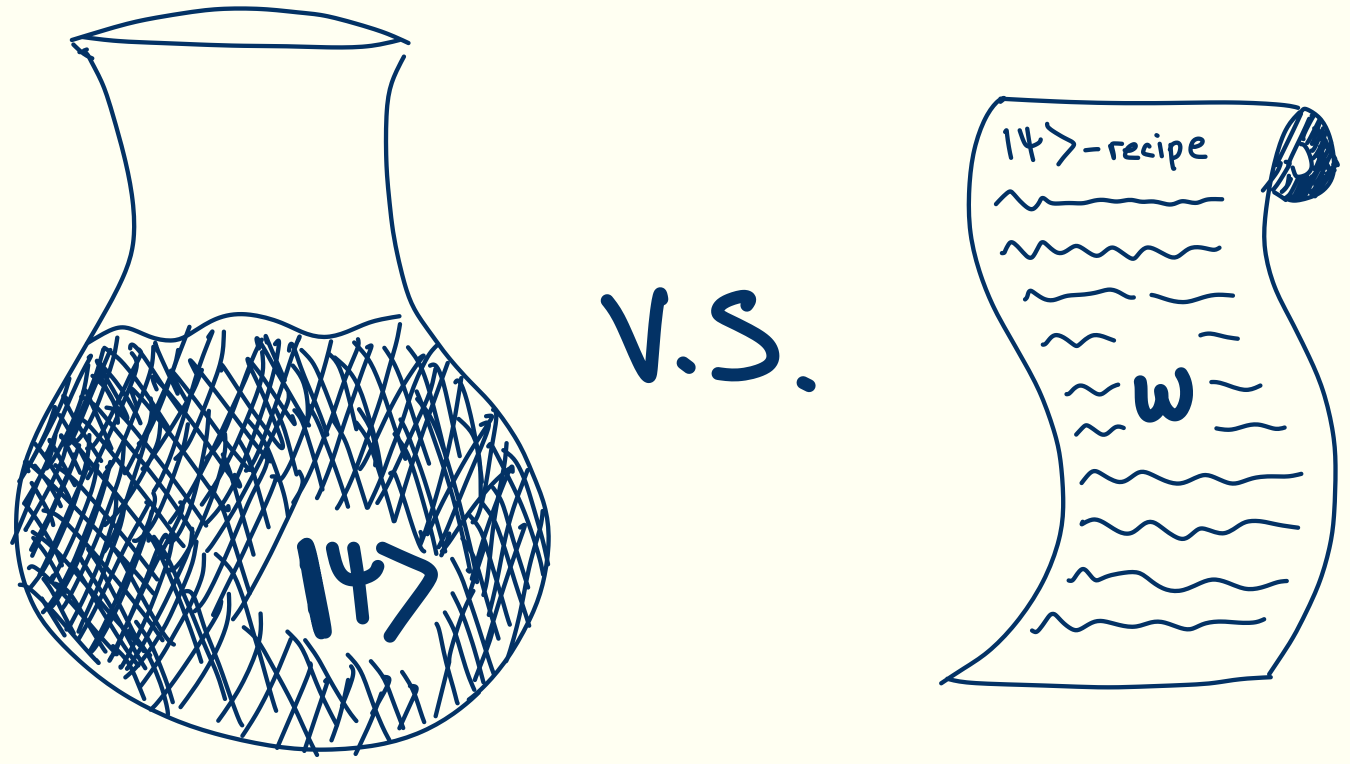
- For this problem, a classical oracle separation is much more challenging (and hopefully interesting) than a quantum oracle separation.

We prove that there is a classical oracle relative to which QMA ≠ QCMA



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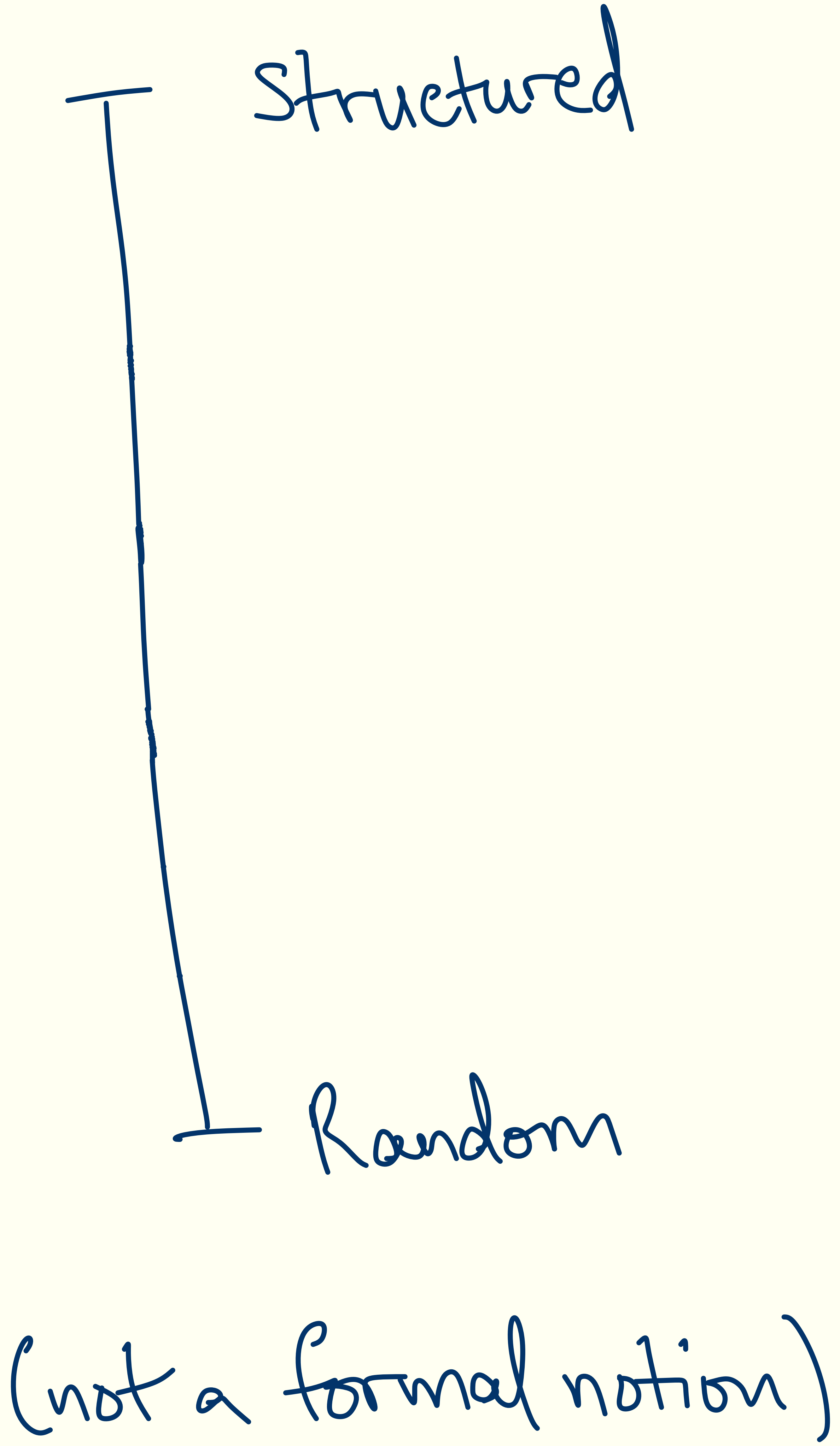
Why is this problem so hard?

To me, the problem has been "stuck" in between two competing desires for a while.



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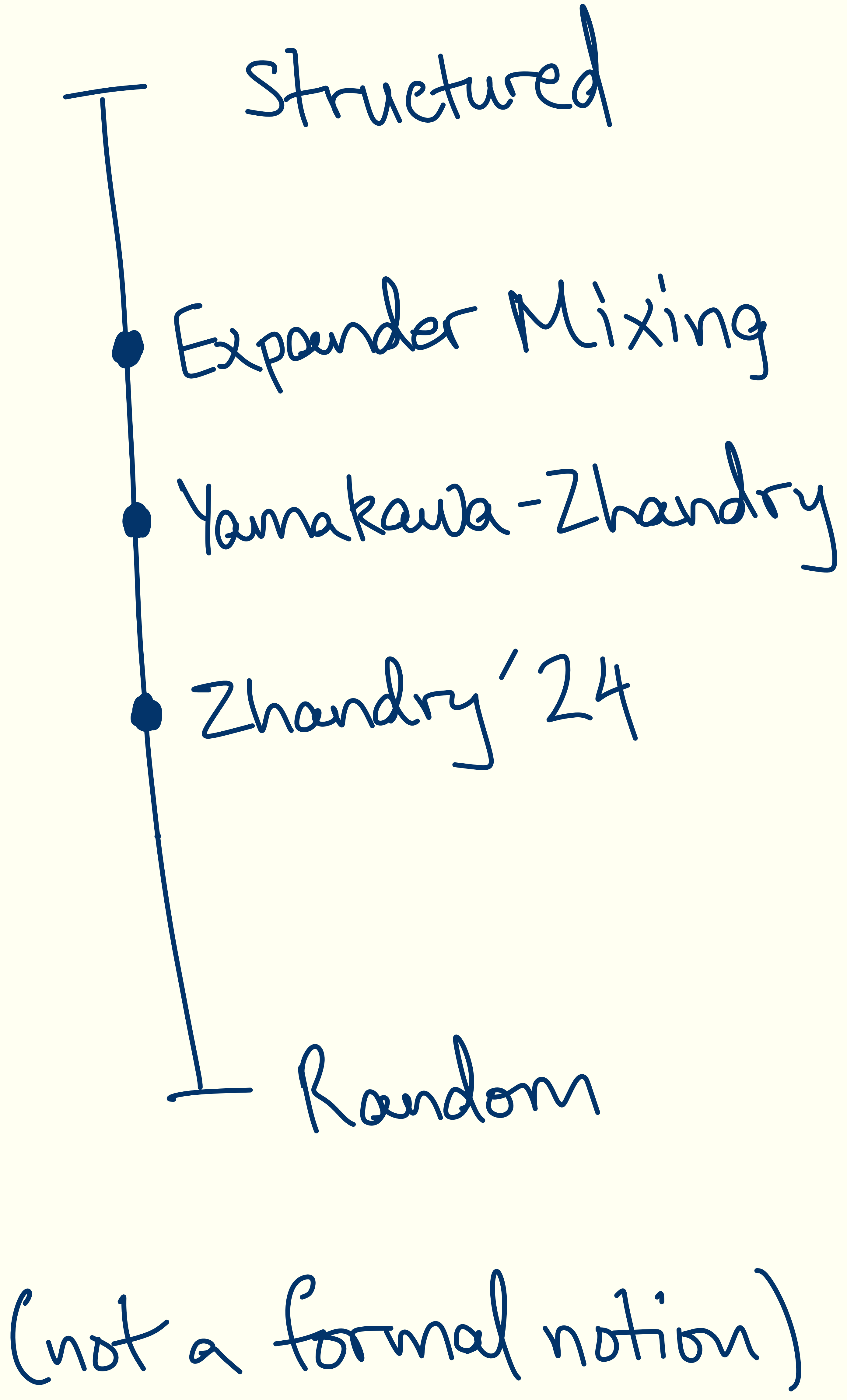

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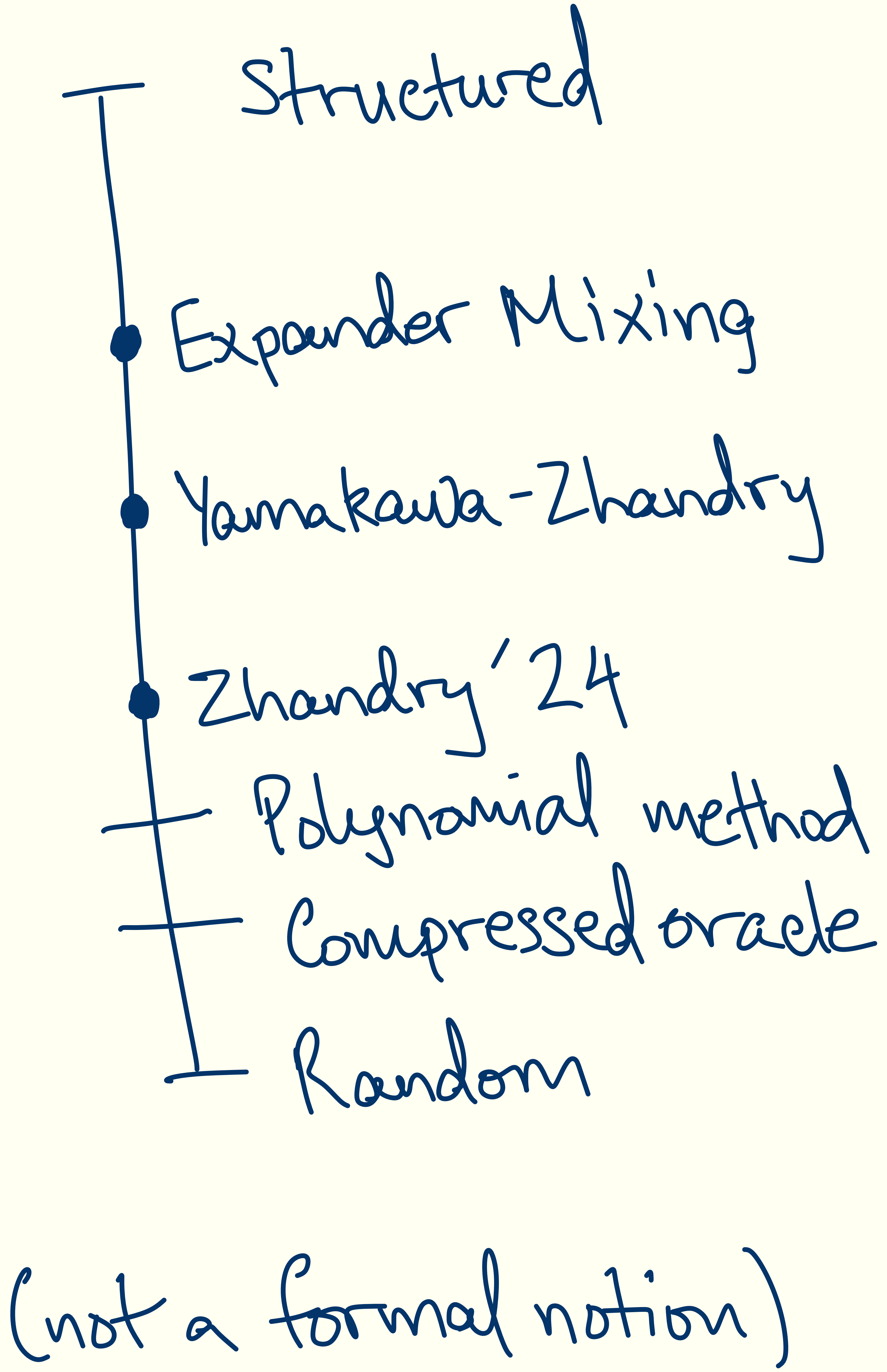

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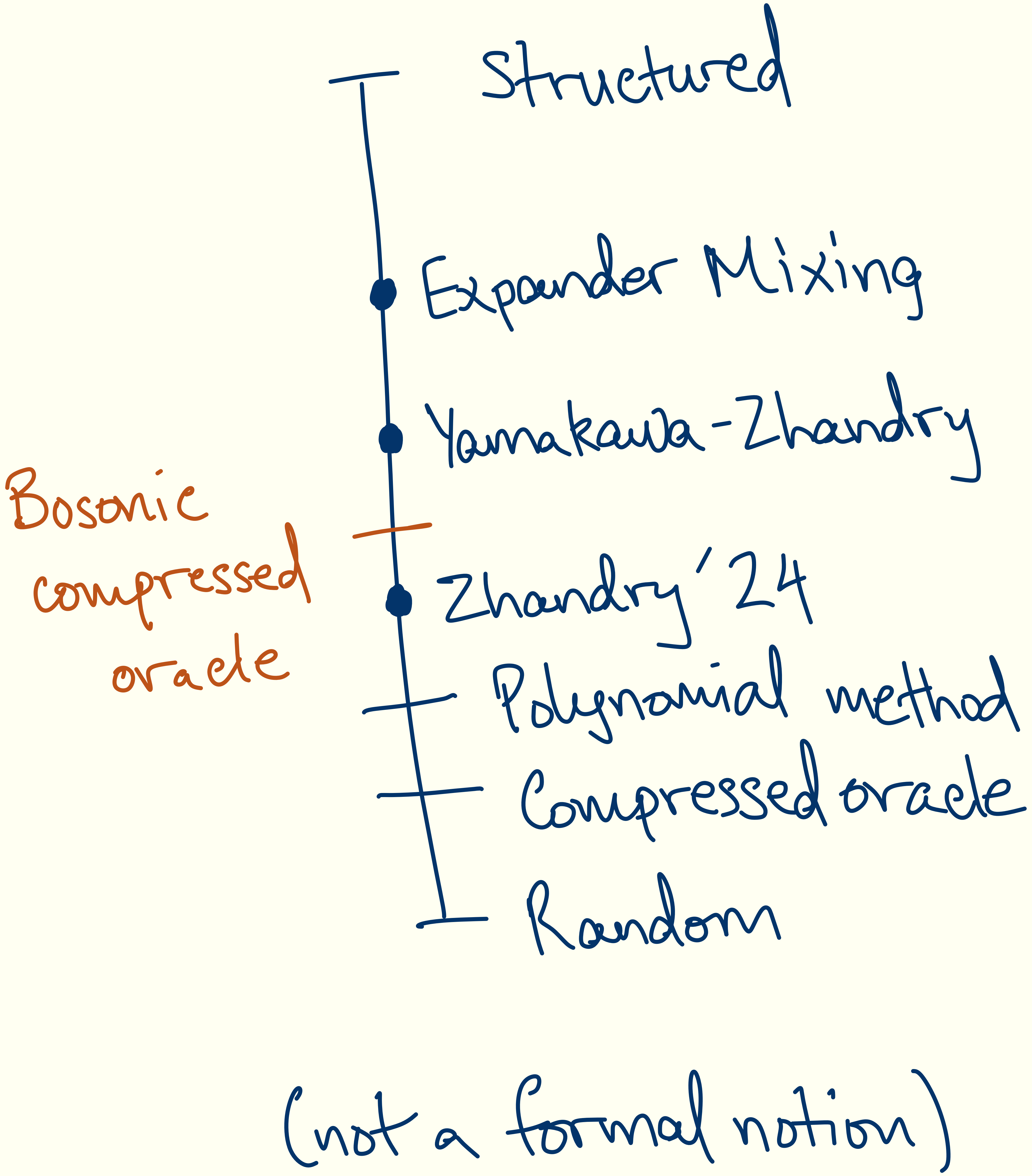
Our paper bridges the gap, taking the less structured oracle of Zhandry'24, and introducing new analysis.



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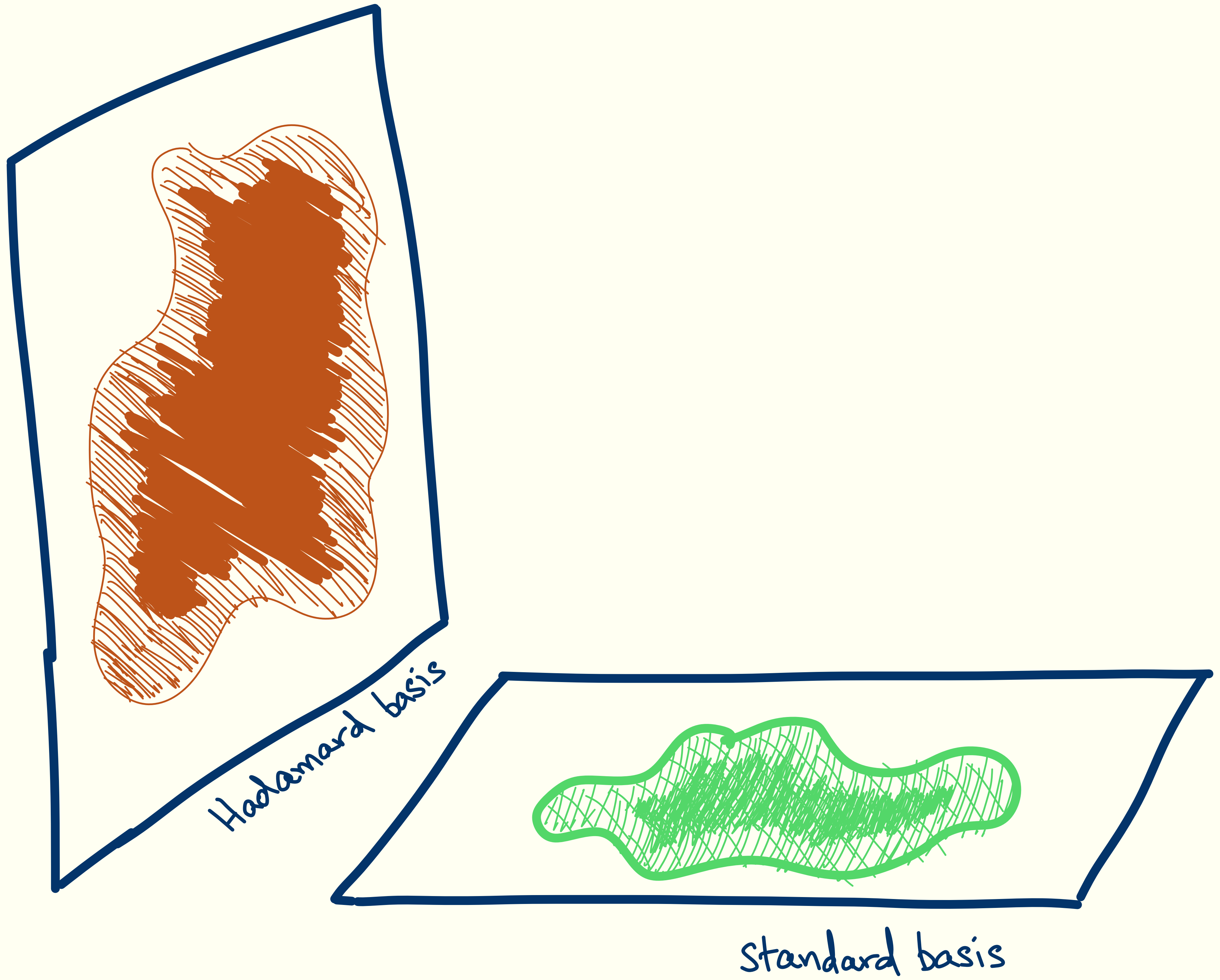
Similarly we can define a scaled-down version of a QCMA verifier.

If you can prove that there is a language separating scaled-down QMA from QCMA, you can use standard diagonalization tricks to turn this into a classical oracle separation.

The spectral Forrelation problem

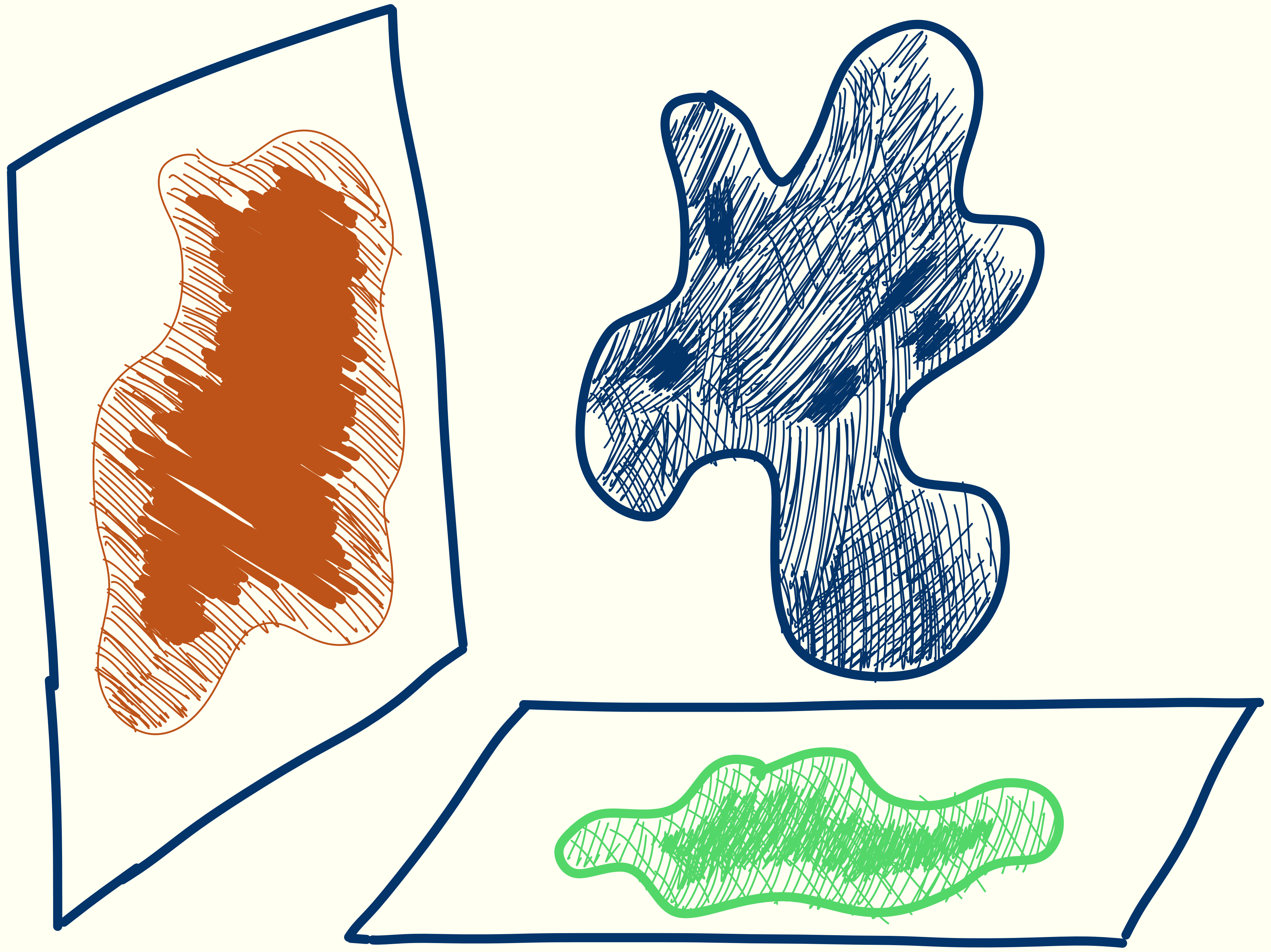
The spectral Forrelation problem is a problem about pairs of sets (S, U), which we treat as oracles through the set membership functions. S~positions, and U~momentums.



The spectral Forrelation problem

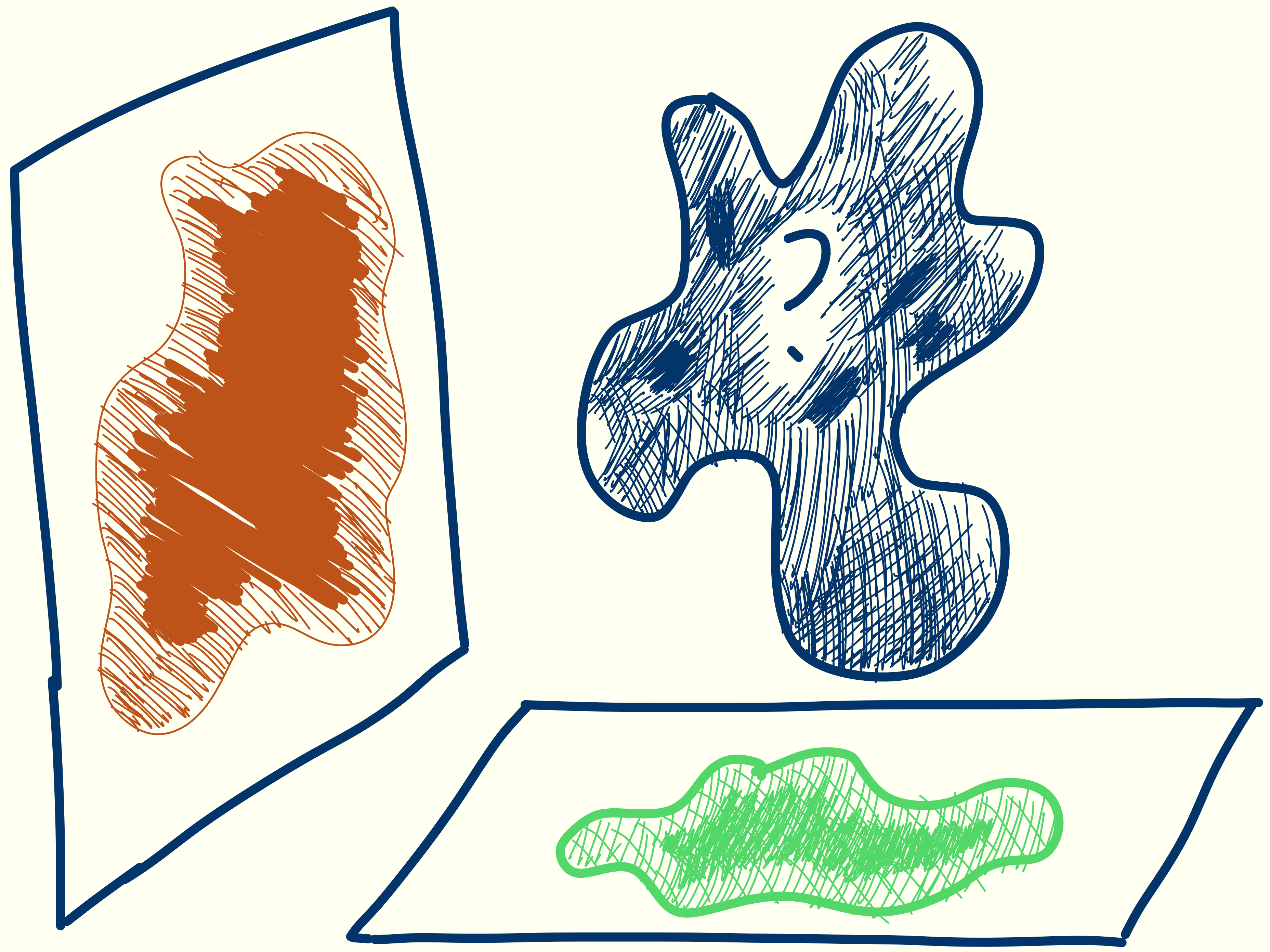
We say that two sets (S, U) are α-spectrally Forrelated if there is a state $|\psi\rangle$ such that

$$\|\Pi_U \cdot H^{\otimes n} \cdot \Pi_S |\psi\rangle\|^2 \geq \alpha$$



The spectral Forrelation problem

Given oracle access to two sets (S, U) (via set membership functions), determine if there is a state $|\psi\rangle$ such that $\|\Pi_U \cdot H^{\otimes n} \cdot \Pi_S |\psi\rangle\|^2$ is large ($\geq 59/100$) or small ($\leq 57/100$), promised that one of the two is the case.



Spectral Forrelation is in QMA

Given a copy of a state $|\psi\rangle$:

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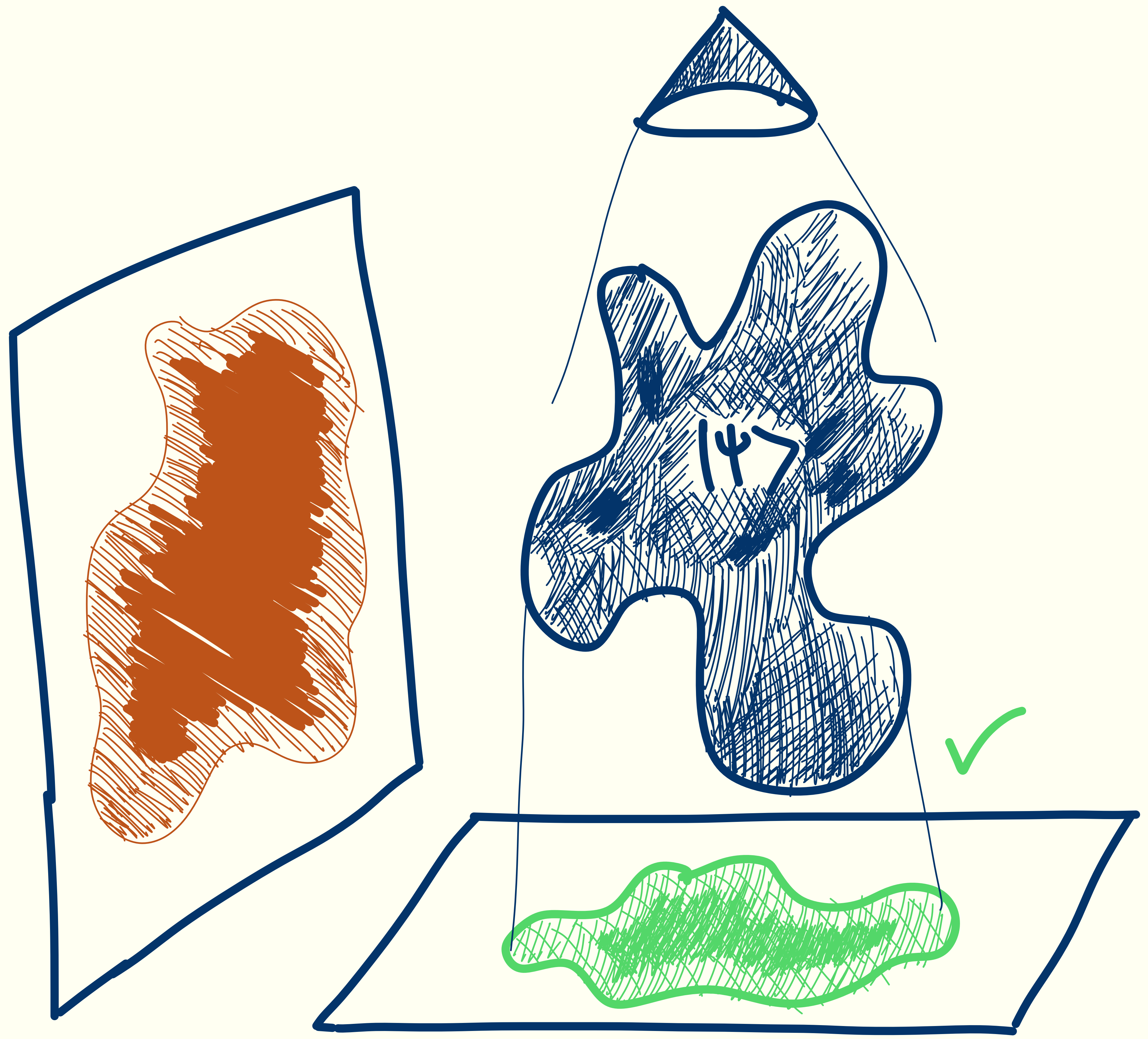

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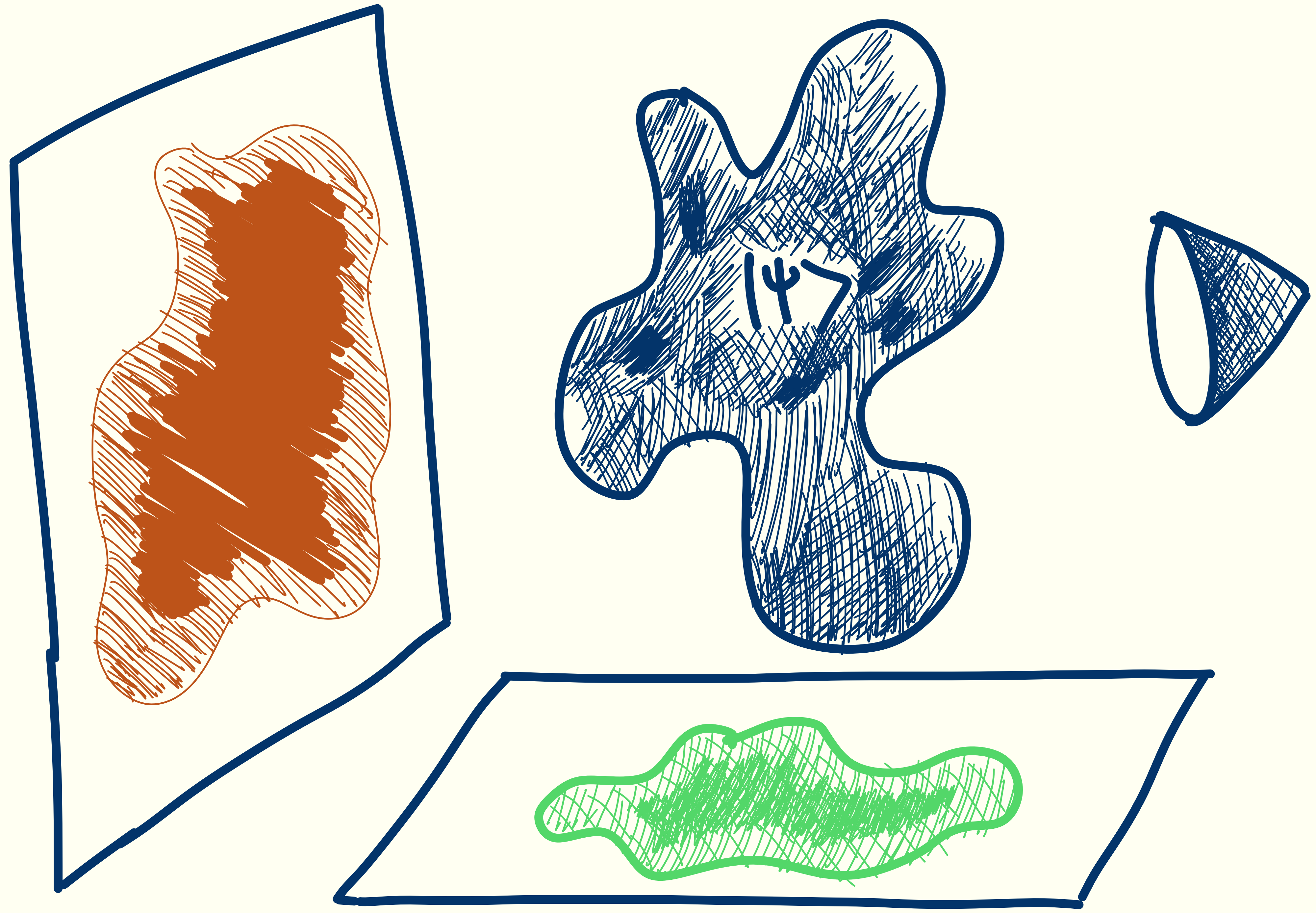
- Apply $H^{\otimes n}$ to the resulting state.



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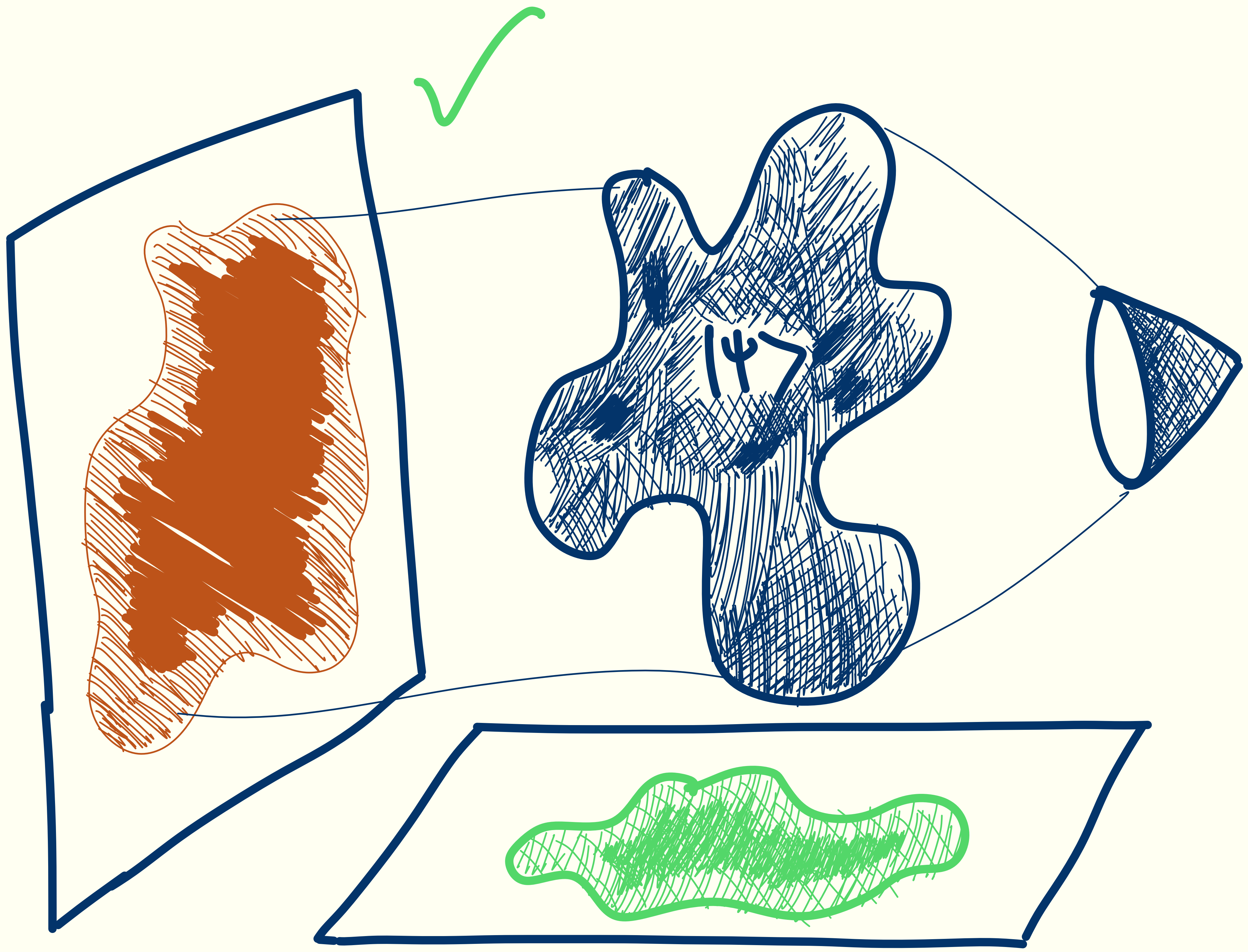
- Accept.



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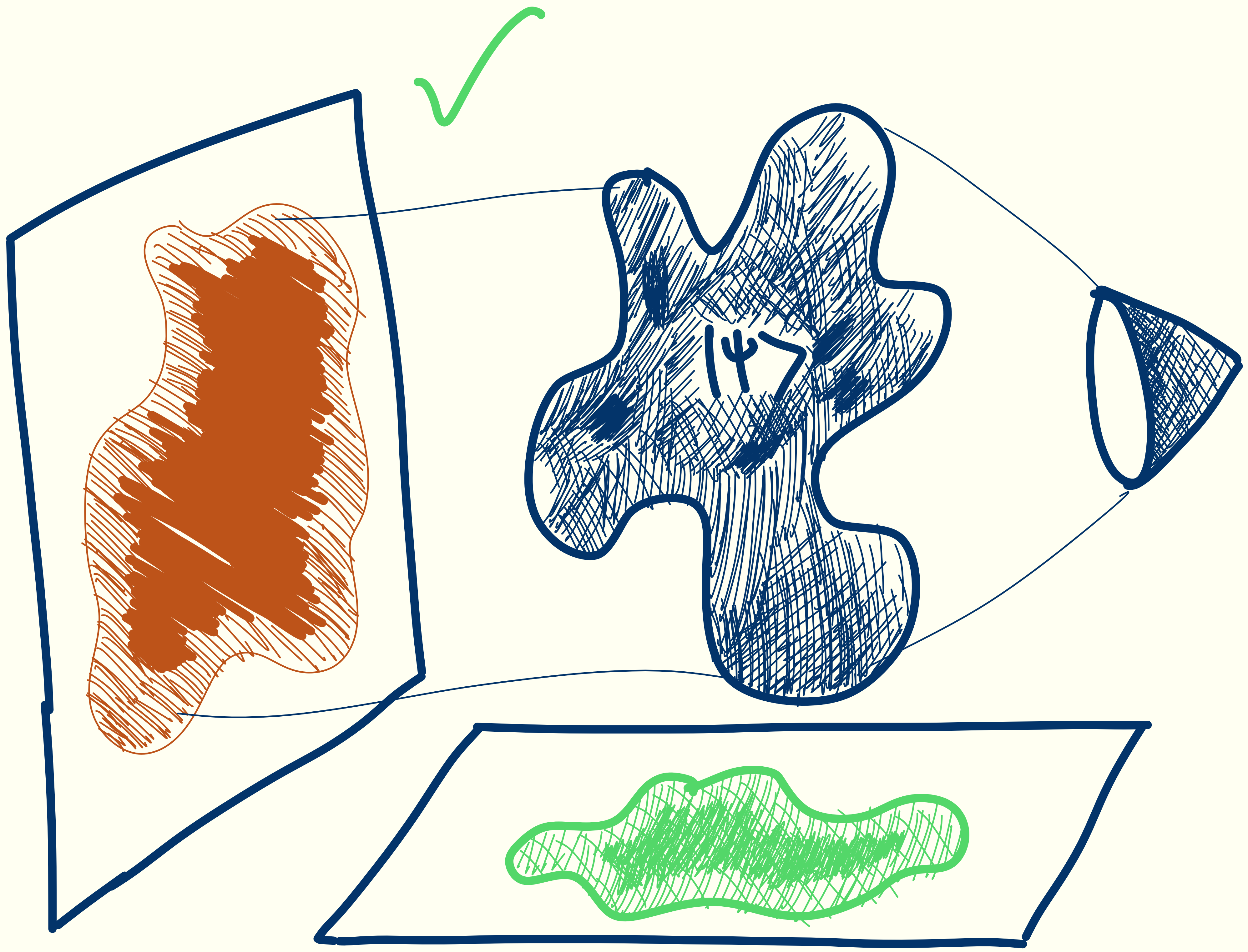
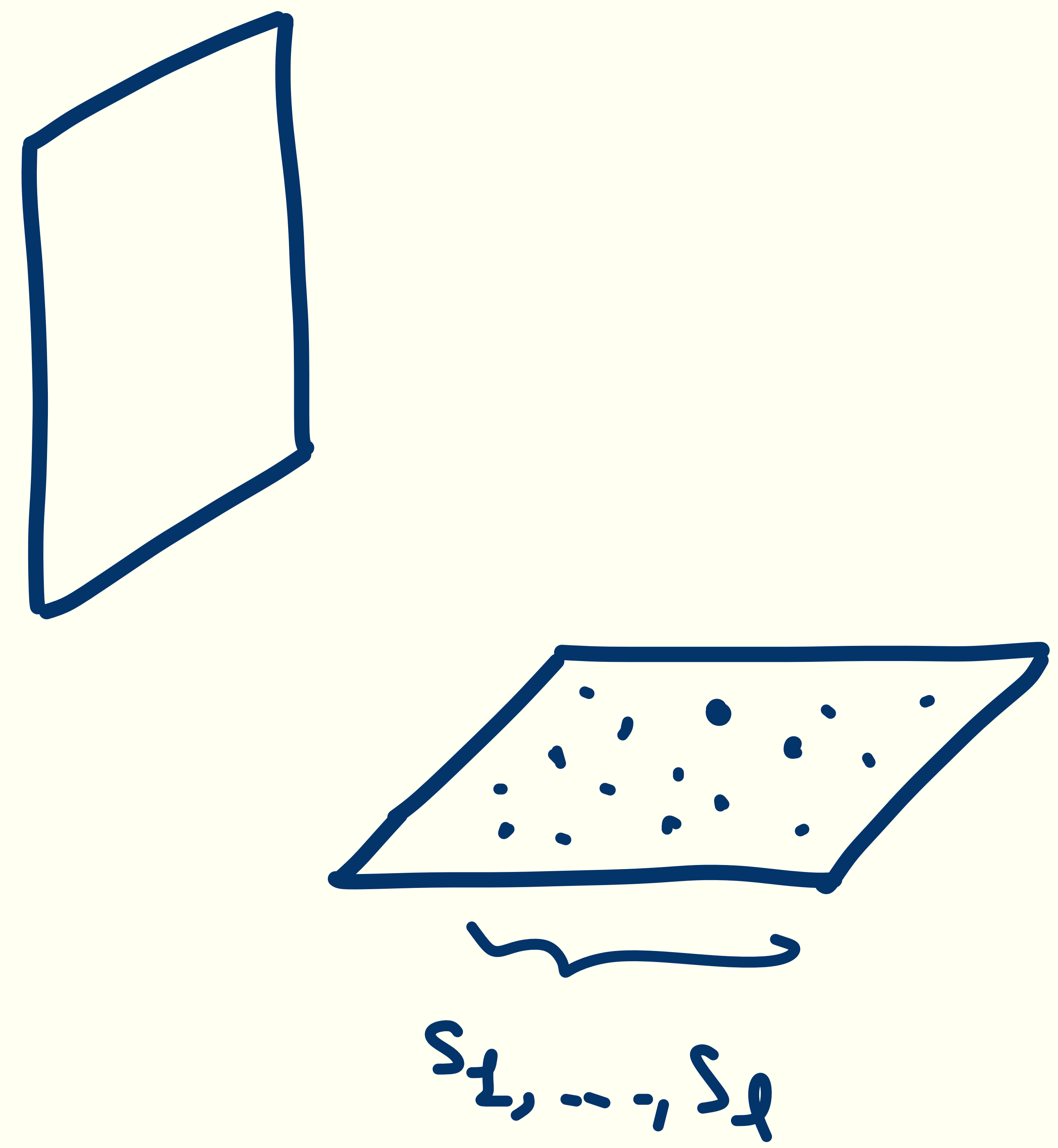


This verifier accepts with probability: $\|\Pi_U \cdot H^{\otimes n} \cdot \Pi_S |\psi\rangle\|^2$.

Marriot-Watrous amplification can bring this to the standard 2/3 or 1/3.

A distribution over spectral Forrelation yes instances

- We will first sample $\ell = 2^{n/10}$ many random elements $s_1, \ldots, s_\ell$.



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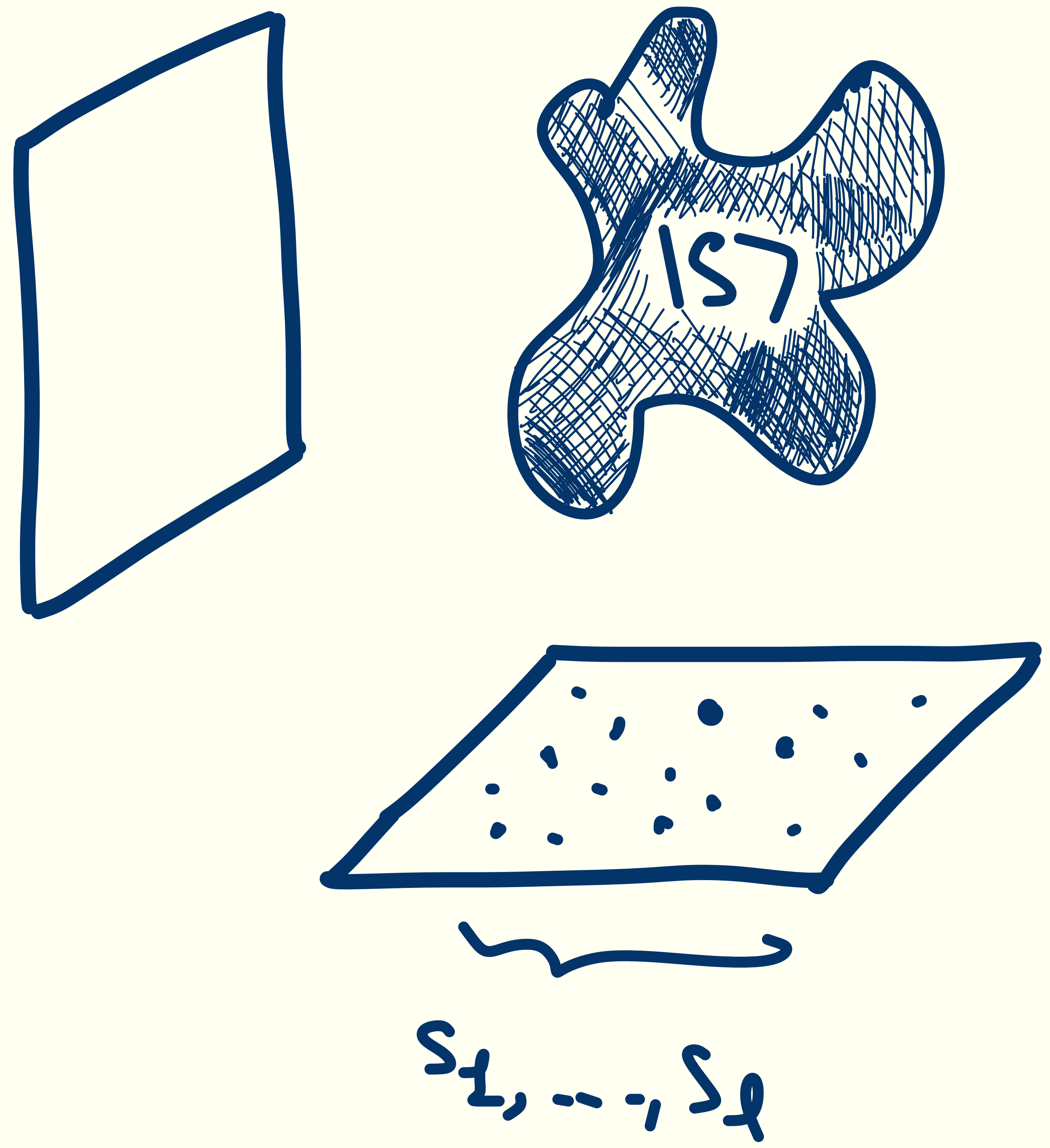

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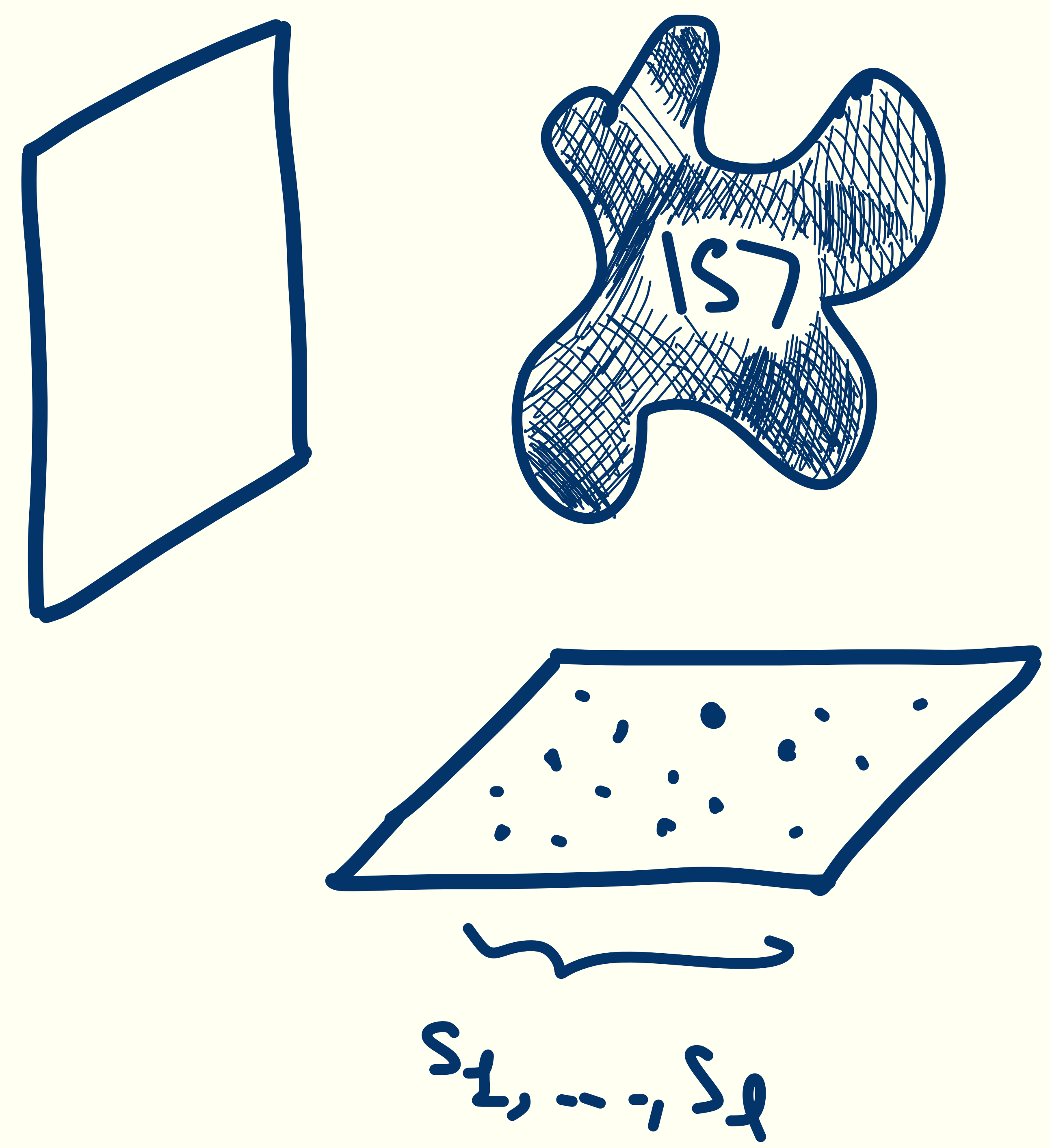
- We take U to be the heavy points of $H^{\otimes n}|S\rangle$, the Hadamard transform of $|S\rangle$:

$$\Pr[y \in U] = 1 - \frac{1}{2}\exp\left(-\frac{1}{10}2^n \left|\langle y|H^{\otimes n}|S\rangle\right|^2\right)$$



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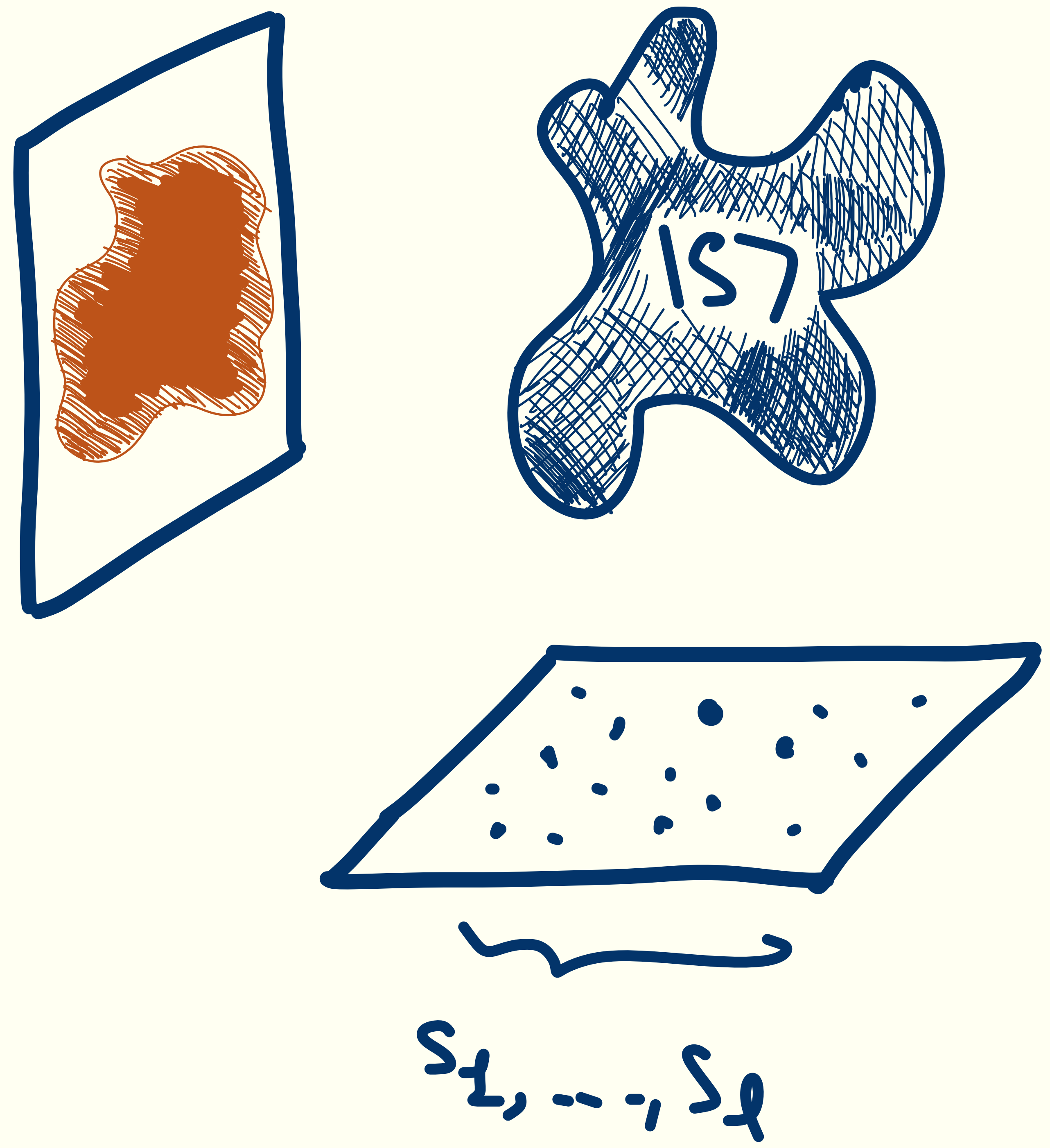

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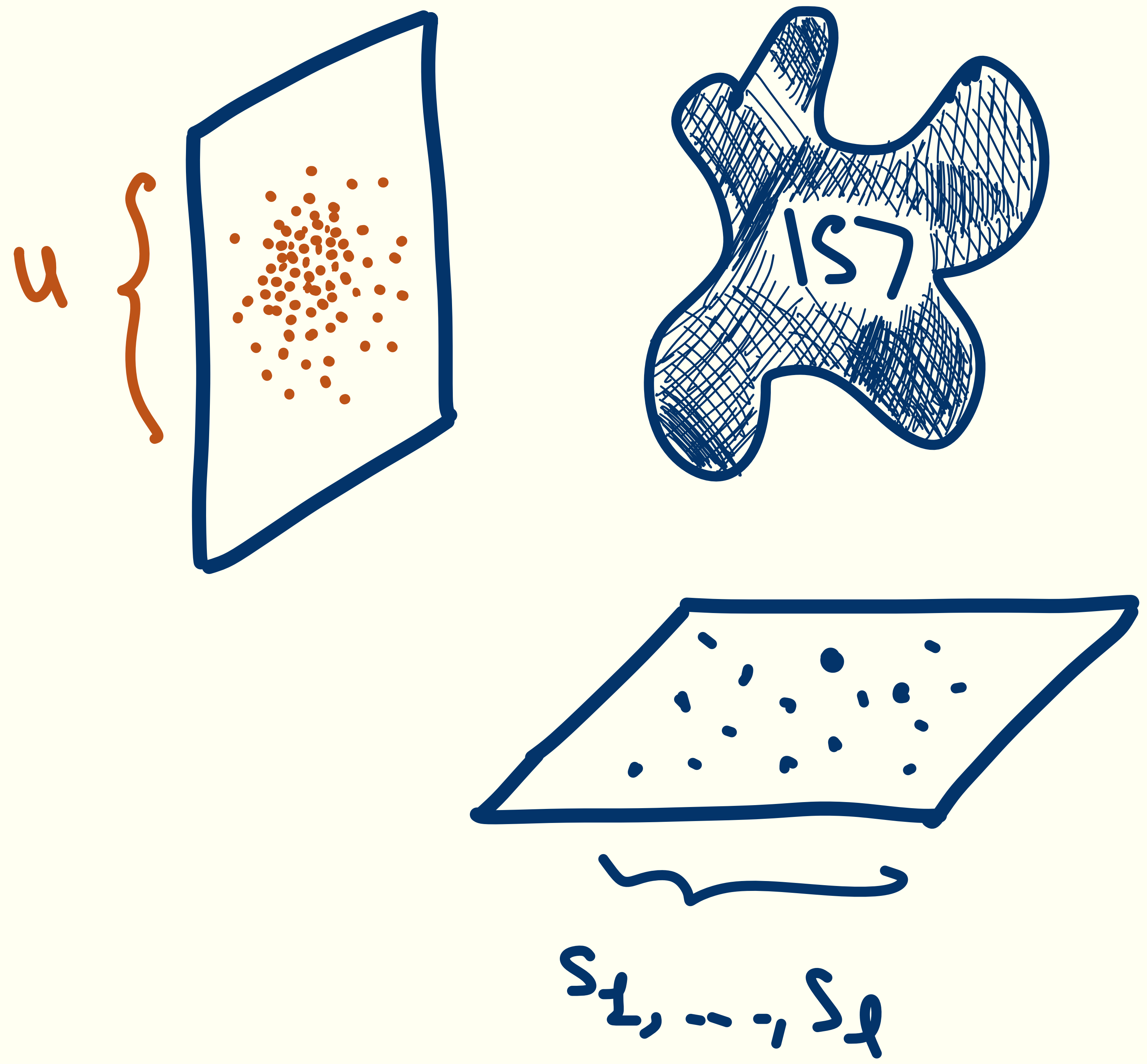

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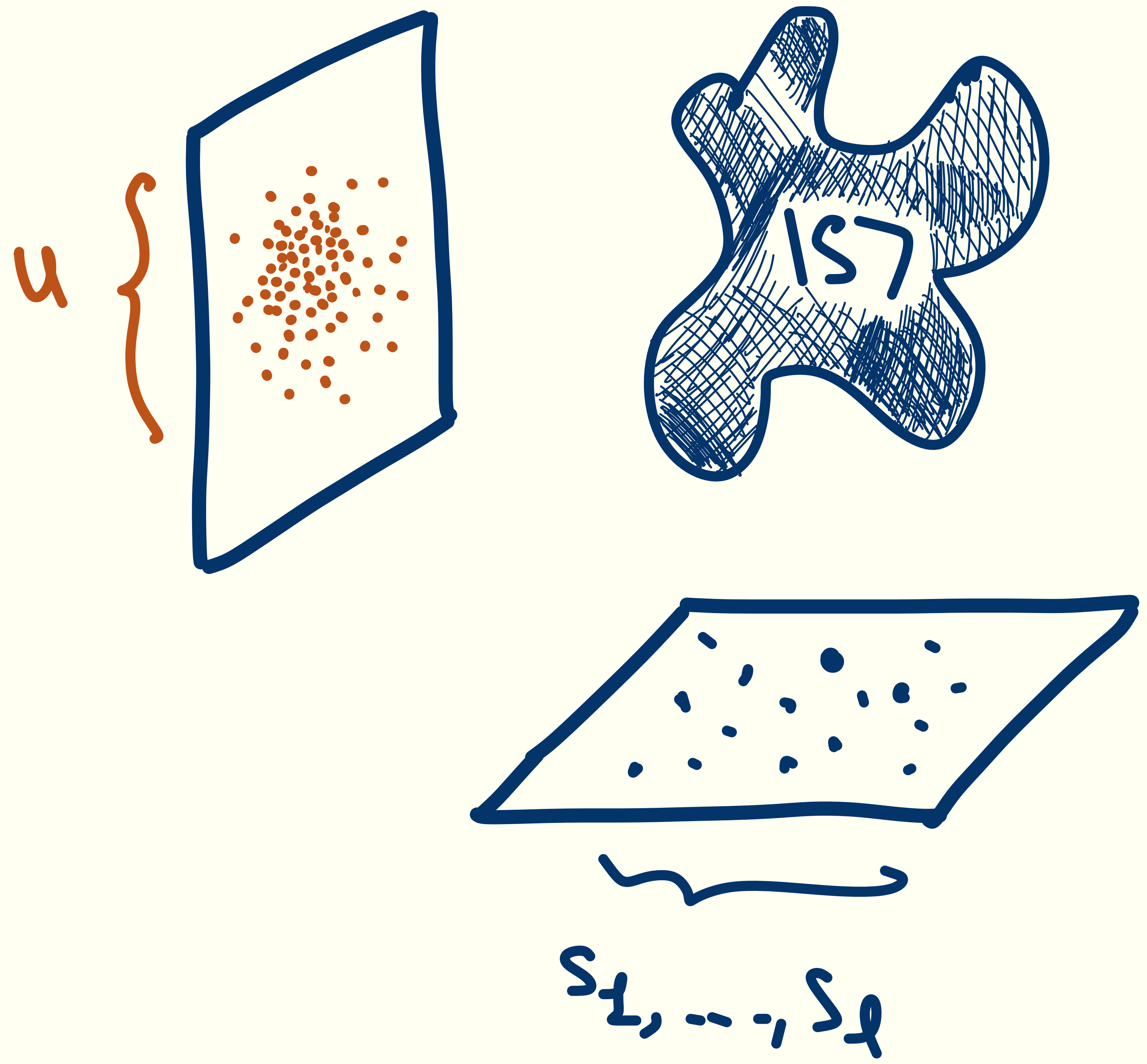

We call this distribution over oracles the Strong distribution.

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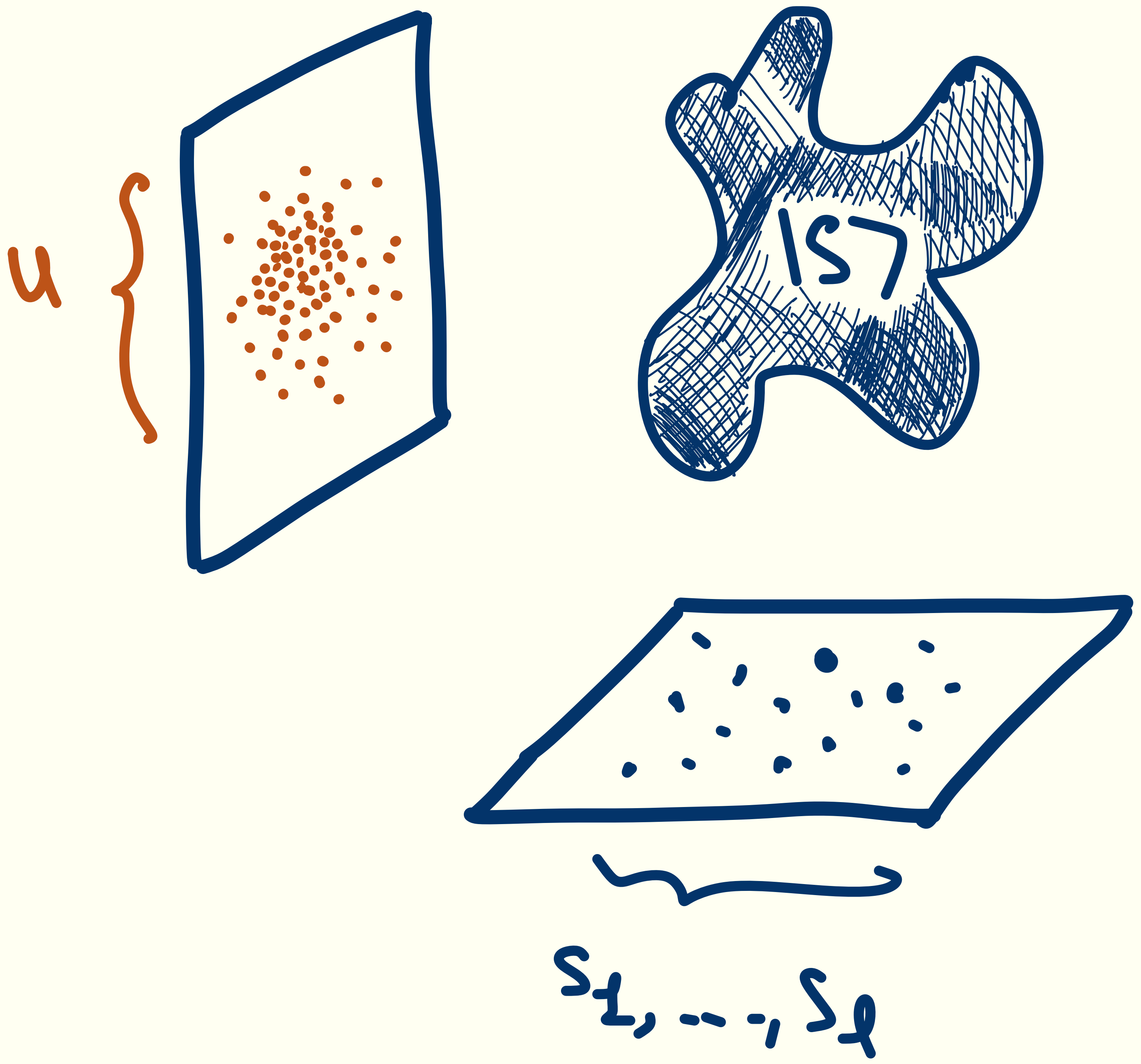

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Theorem 2: If there exists a QCMA algorithm, making $t = t(n)$ queries to (S, U) and taking a witness of length $q = q(n)$, then for all $0 < v < \ell/100$, there is a query algorithm making vt queries to U that outputs v distinct points from S with probability

$$\geq 2^{-q}\left(\frac{1}{36t^2}\right)^v$$

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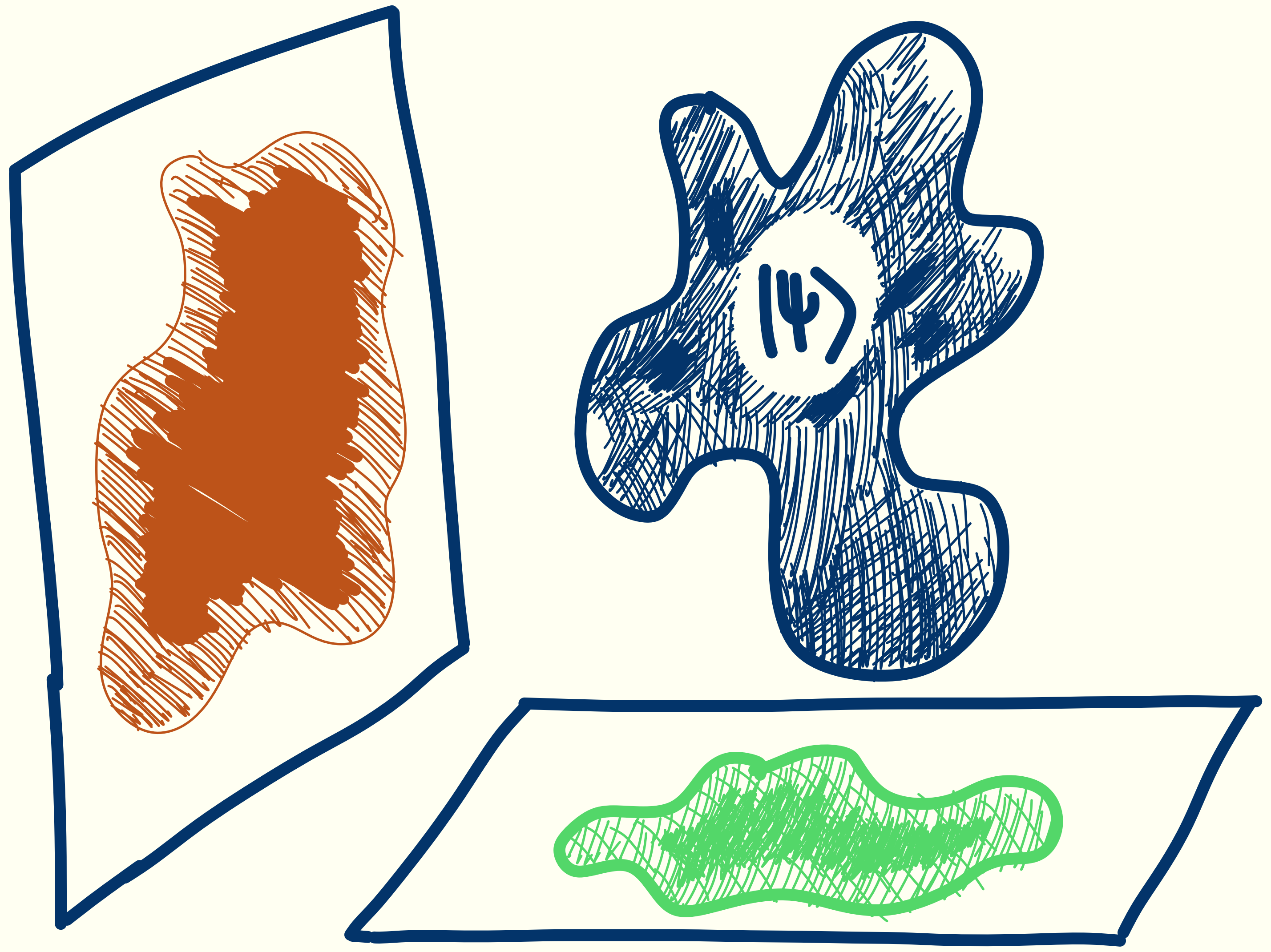

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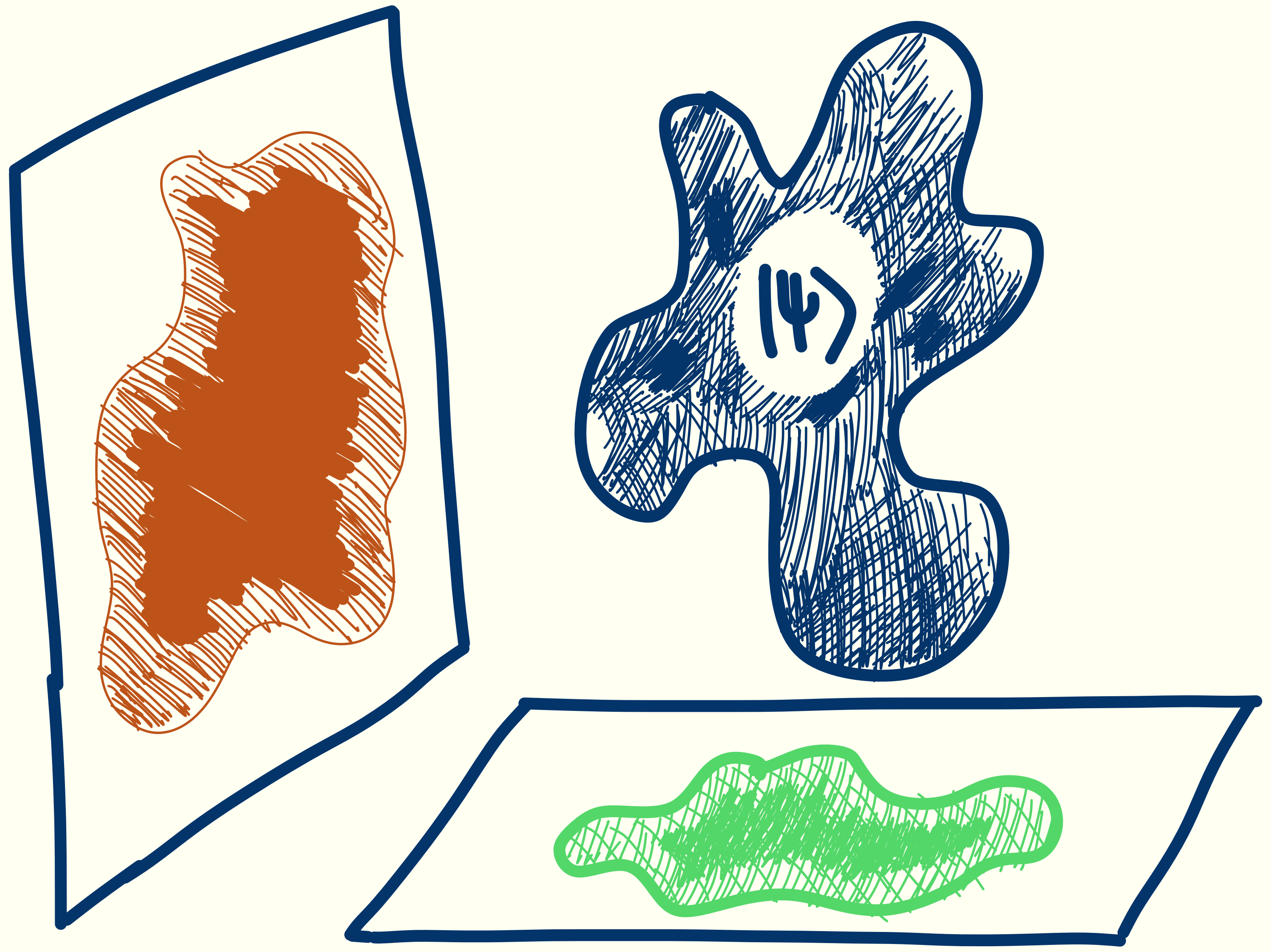
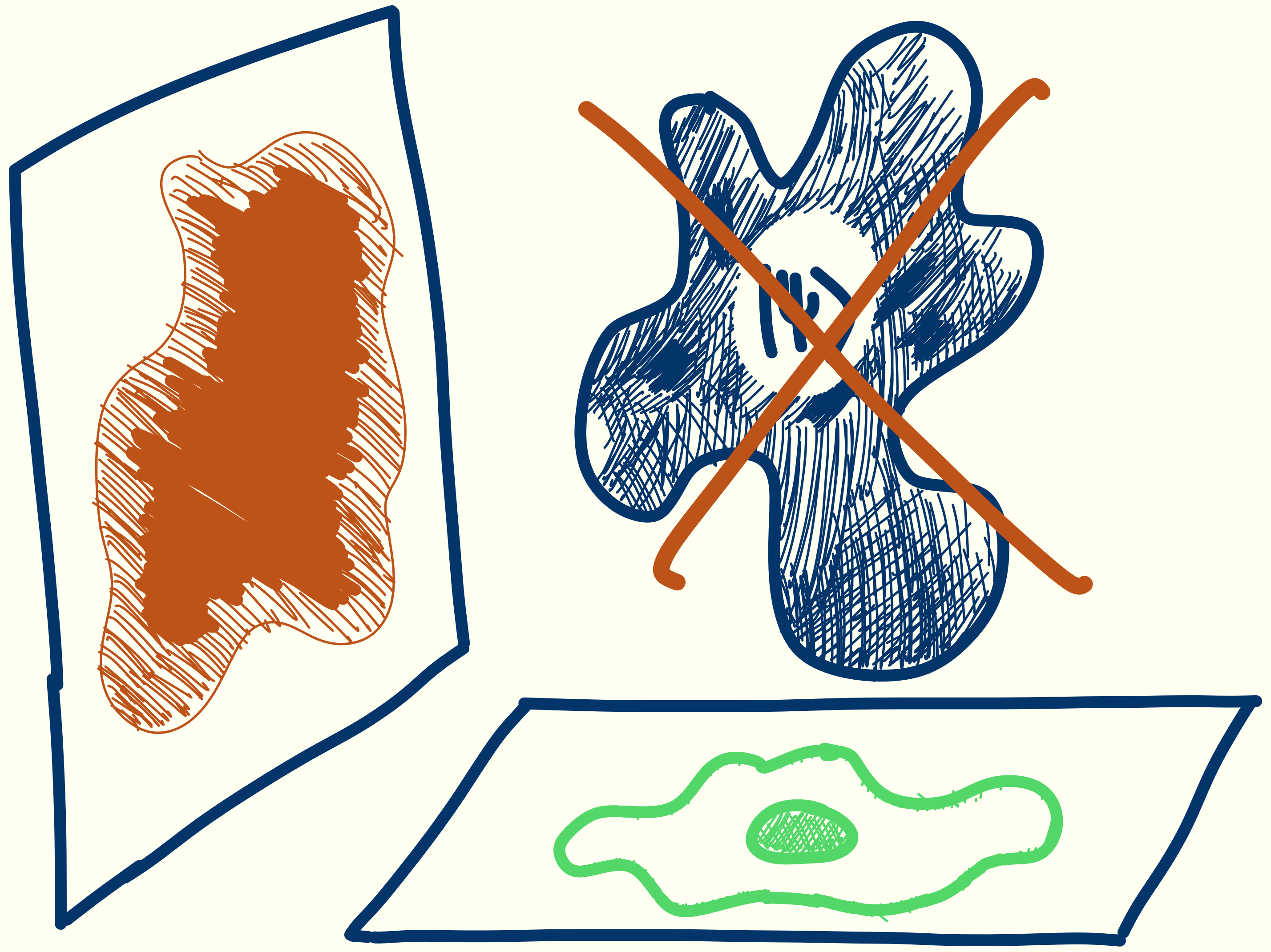
- (S, U) is a yes instance of spectral Forrelation (i.e., $\geq 59/100$ spectrally Forrelated).
- For all $\Delta \subset S$ with $|\Delta| \leq \ell/100$, (Δ, U) is a no instance of spectral Forrelation (i.e., $\leq 57/100$ spectrally Forrelated).





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Proof sketch: When we compute the expectation over U of the "Forrelation" matrix, we roughly get something that looks like

$$\mathbb{E}_U[\Pi_S \cdot H^{\otimes n} \cdot \Pi_U \cdot H^{\otimes n} \cdot \Pi_S] \approx \frac{1}{10} | S \rangle\langle S | + \frac{1}{2}\mathrm{id}$$

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Proof sketch: When we compute the expectation over U of the "Forrelation" matrix, we roughly get something that looks like

$$\mathbb{E}_U[\Pi_S \cdot H^{\otimes n} \cdot \Pi_U \cdot H^{\otimes n} \cdot \Pi_S] \approx \frac{1}{10} |S\rangle\langle S| + \frac{1}{2}\mathrm{id}$$

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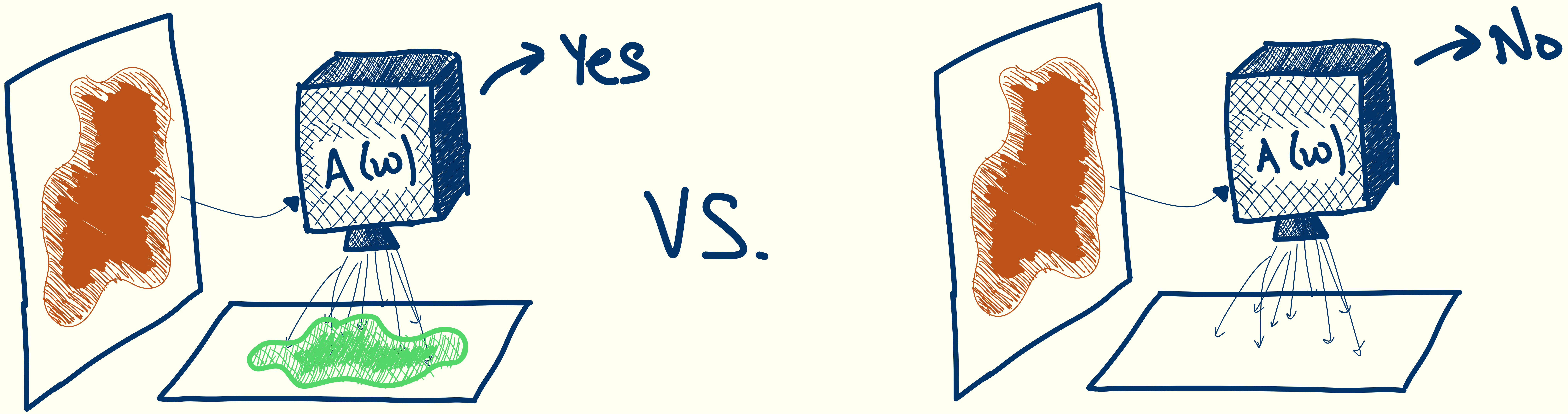
Using concentration bounds, we get that with very high probability, this happens.

Strong yes instances can be sampled from

Any quantum query algorithm that distinguishes between (S, U) and $(\varnothing, U)$ must query a point in S pretty often ($\geq 1/3t$ chance per query), since otherwise the action of the oracles is identical.



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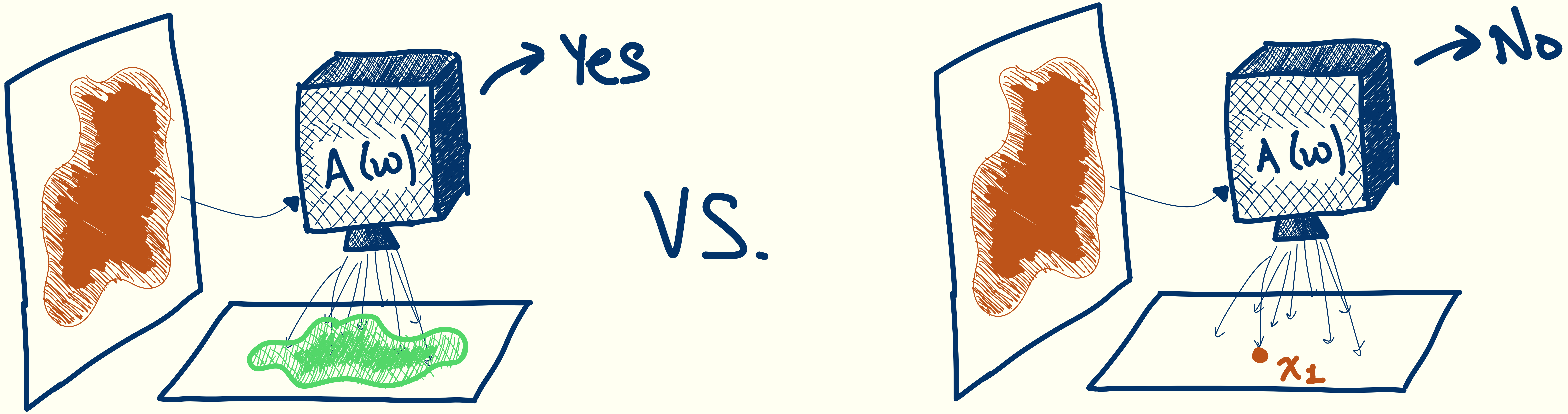

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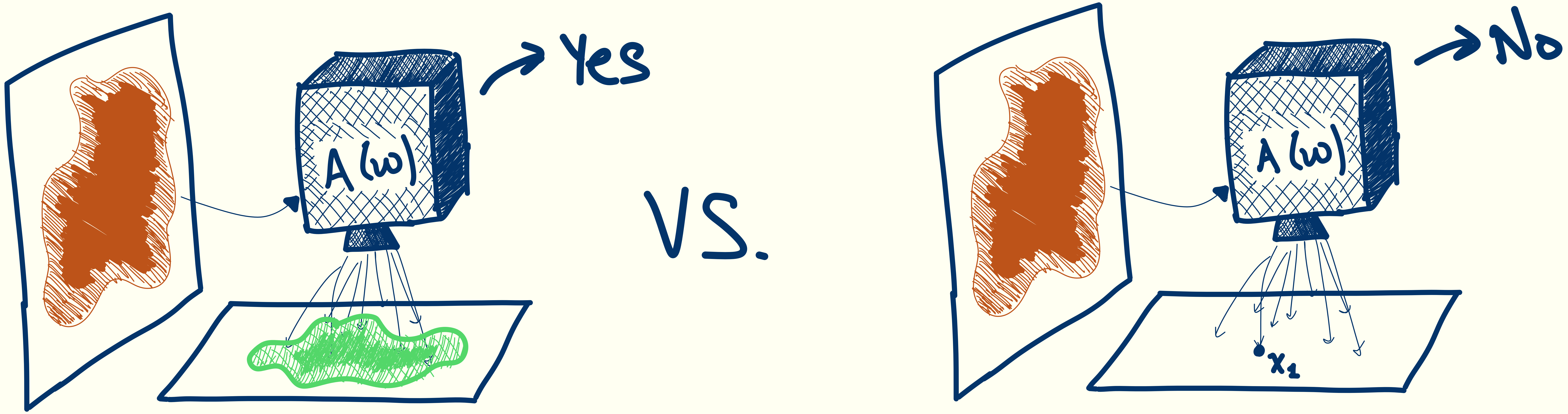

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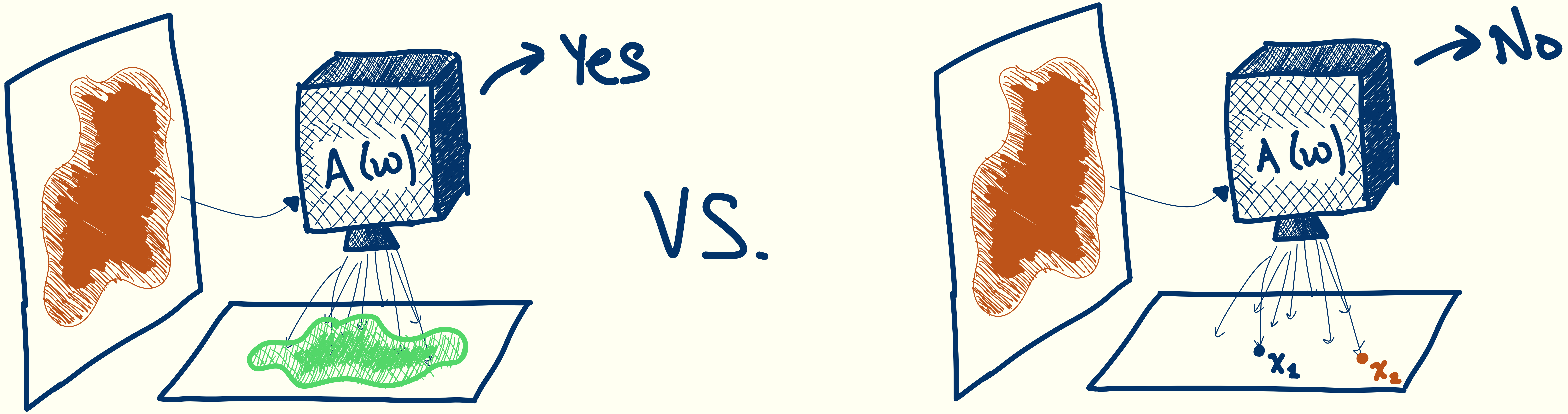
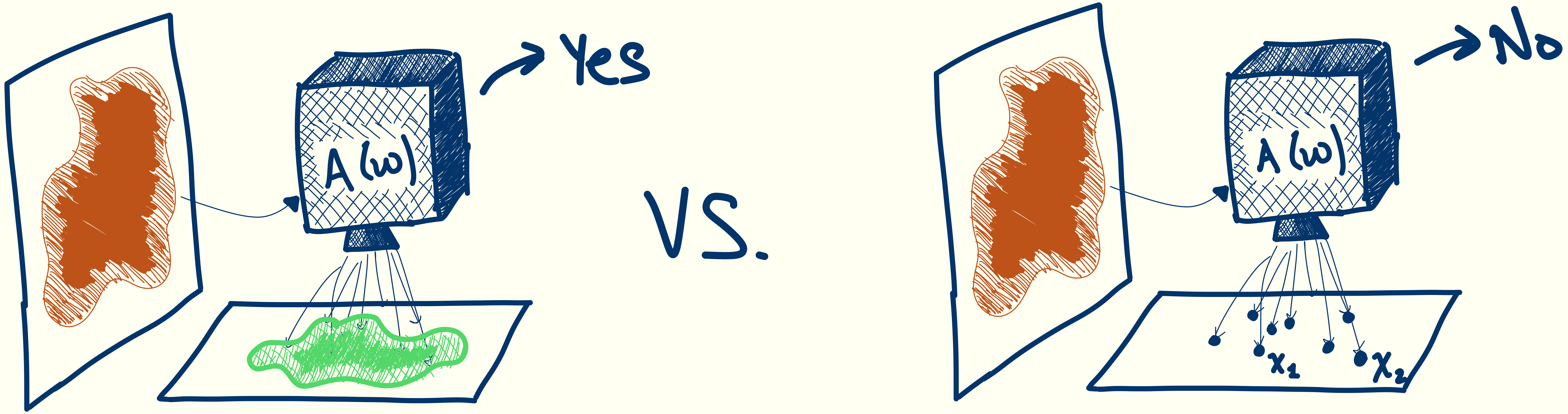


Therefore, measuring a random query of the algorithm will yield a point in S with good probability, x_2.

Strong yes instances can be sampled from

Because of the strong yes property, we can keep going until $\ell/100$ points have been sampled! This is the key step that uses the fact that the witness is classical.



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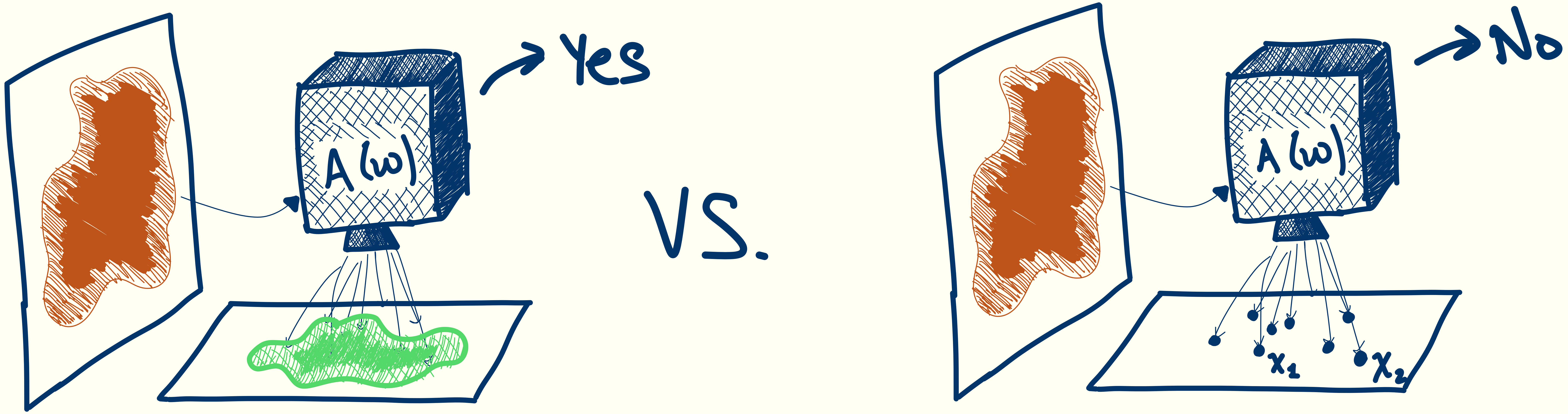

Given a QCMA algorithm, we can guess the classical witness and be correct with probability 2^{-q}.

Rest of the talk: sampling probability upper bound via the **compressed oracle technique**.

Recall: The theorem we're trying to prove:

Theorem 1: For all $v > 0$, and all quantum query algorithms making $T = T(n)$ queries to a set membership oracle for U, the probability, over Strong, that the algorithm outputs v distinct points from S is at most

$$\leq \left(\frac{\mathrm{poly}(v, T)}{\mathrm{poly}(2^n)}\right)^v.$$

Purification of quantum query algorithms

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$$\prod_{i=t}^{1}\left(\mathsf{pfO}\cdot A\right)|0\rangle\sum_{\mathcal{O}}\sqrt{D(\mathcal{O})}\,|\mathcal{O}\rangle$$

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The purifying register starts out unentangled and a successful algorithm will generate a specific type of entanglement (depending on the task)

Compressed oracles for Strong

Recall that we sampled Strong via the following:

- We will first sample $\ell = 2^{n/10}$ many random elements $s_1, \ldots, s_\ell$. Let $|S\rangle$ be the uniform superposition over the points.

- We take U to be the heavy points of $H^{\otimes n}|S\rangle$, the Hadamard transform of $|S\rangle$:

$$\Pr[y \in U] = 1 - \frac{1}{2}\exp\left(-\frac{1}{10}\left|\langle y|H^{\otimes n}|S\rangle\right|^2\right)$$

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Let's focus on purifying the distribution over S first!

Compressed oracles for Strong

The Fock basis is a way to write down a multi-set, similar to how we write subsets of $\{0,1\}^n$ as 2^n bit strings. Given a multi-set with ℓ_x copies of x, we associate it with a vector of 2^n non-negative integers:

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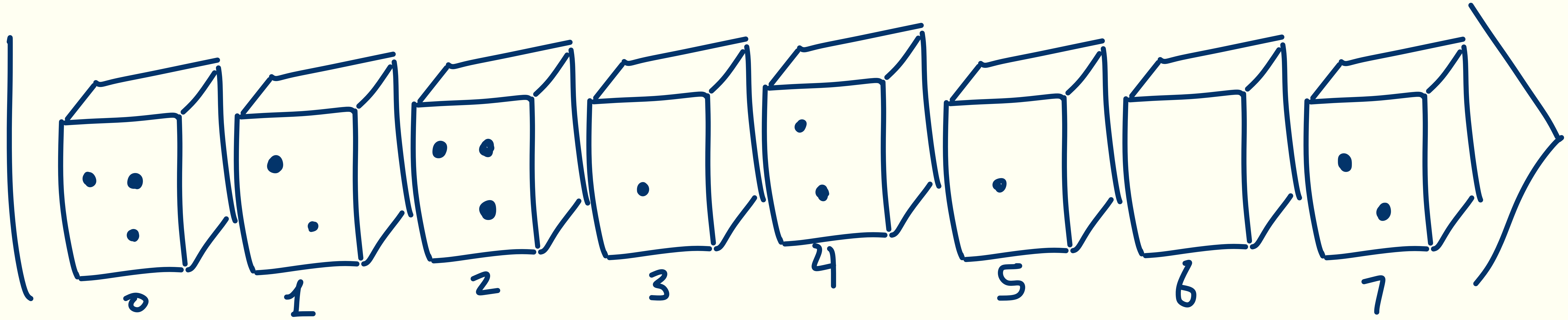
Then the uniform superposition over multi-sets is given by

$$\frac{1}{\sqrt{2^n}} \sum_{\vec{\ell}} \sqrt{\frac{\ell!}{\prod_x \ell_x!}} |\ell_0, \ldots, \ell_{2^n}\rangle .$$

Compressed oracles for Strong: Bosons

Bosons are a mathematical representation of multi-sets used in physics. The "annihilation" and "creation" operators add and subtract elements (bosons).



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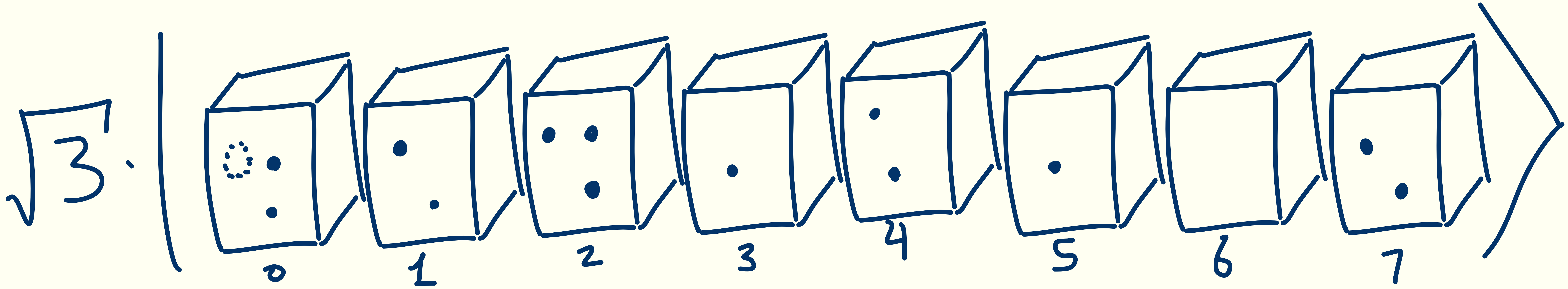

$$\hat{a}_x|\ell_0, \ldots, \ell_x, \ldots, \ell_{2^n-1}\rangle = \sqrt{\ell_x}|\ell_0, \ldots, \ell_x - 1, \ldots, \ell_{2^n-1}\rangle$$

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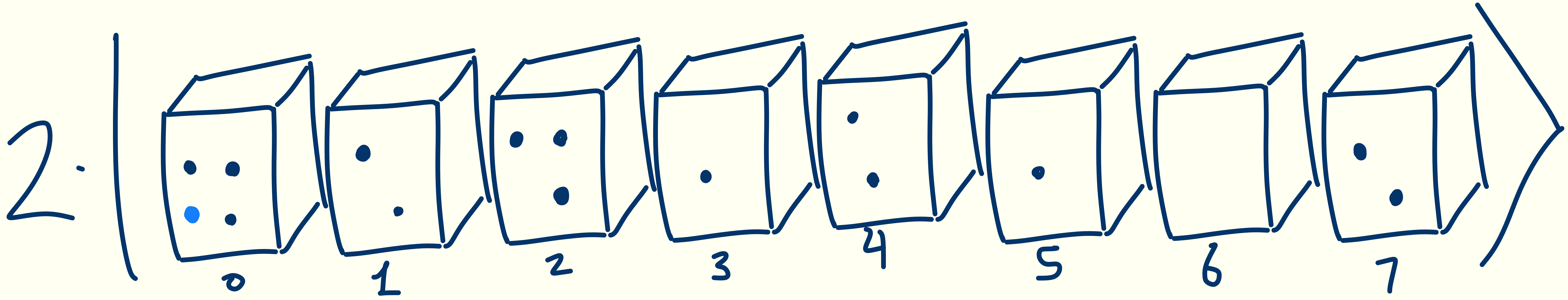

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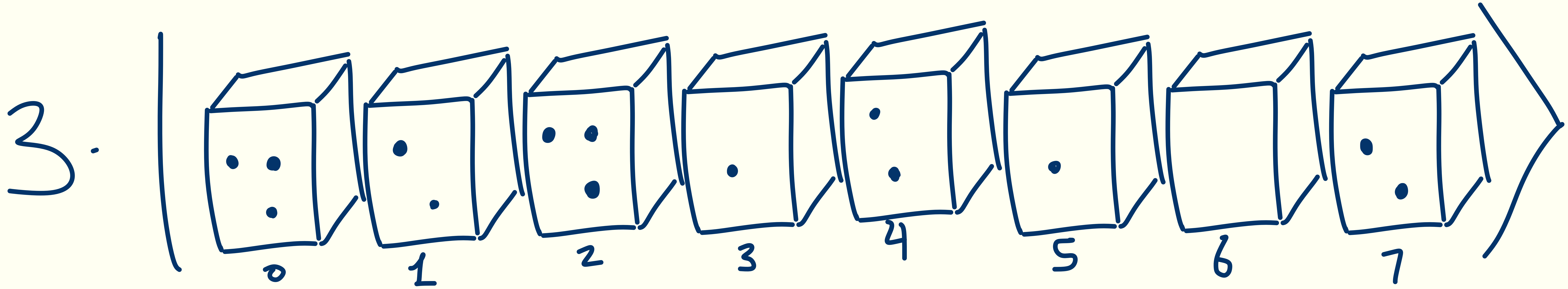

$$\hat{a}_x |\ell_0, \ldots, \ell_x, \ldots, \ell_{2^n-1}\rangle = \sqrt{\ell_x} |\ell_0, \ldots, \ell_x - 1, \ldots, \ell_{2^n-1}\rangle$$

$$\hat{a}_x^\dagger |\ell_0, \ldots, \ell_x, \ldots, \ell_{2^n-1}\rangle = \sqrt{\ell_x + 1} |\ell_0, \ldots, \ell_x + 1, \ldots, \ell_{2^n-1}\rangle$$

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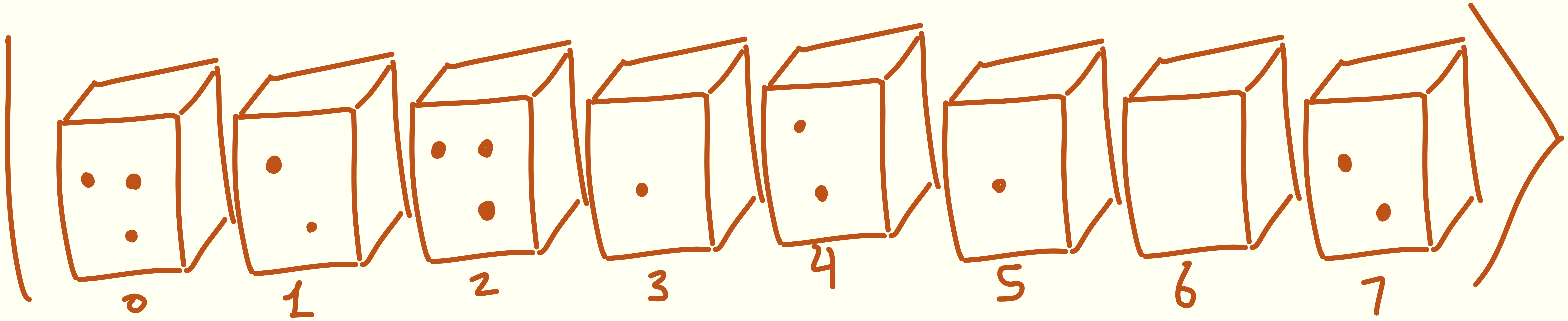
Taking the product, the number operator $\hat{n}_x = \hat{a}_x^\dagger \hat{a}_x$ is diagonal in the position Fock basis, and applies a scaling of the number of bosons in the x'th mode.

Compressed oracles for Strong: Bosons

We can also define a "Hadamard" basis for the bosons, with the analogous operators:

$$\widetilde{a}_y = \frac{1}{\sqrt{2^n}} \sum_x (-1)^{y \cdot x} \hat{a}_x \quad \text{and} \quad \widetilde{a}_y^\dagger = \frac{1}{\sqrt{2^n}} \sum_x (-1)^{y \cdot x} \hat{a}_x^\dagger$$

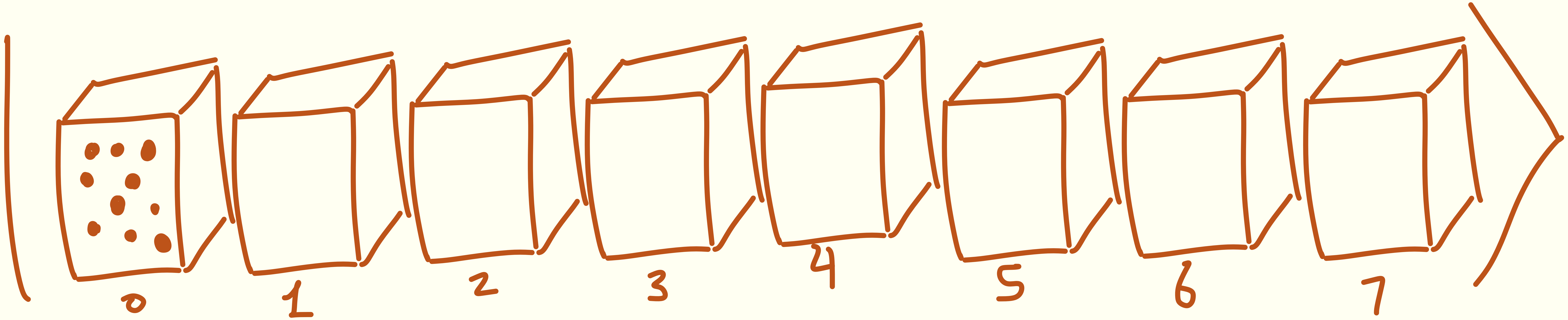


As a matter of notation, we also define $|\text{vac}\rangle$ to be the state $|0,\ldots,0\rangle$.

Compressed oracles for Strong: Bosons

Claim: The purification of a random multi-set is:

$$|\text{init}\rangle = \frac{1}{\sqrt{\ell!}}\left(\tilde{a}_0^\dagger\right)^\ell |\text{vac}\rangle$$



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Proof: Expand out the expression:

$$\frac{1}{\sqrt{\ell!}} \left(\tilde{a}_0^\dagger\right)^\ell |\text{vac}\rangle = \frac{1}{\sqrt{2^{n\ell} \cdot \ell!}} \sum_{s_1,\ldots,s_\ell} \hat{a}_{s_1}^\dagger \ldots \hat{a}_{s_\ell}^\dagger |\text{vac}\rangle$$

If you deal with the coefficients (and multiplicities of the multi-sets), it works out. 😊

Compressed oracles for Strong: Action of U

Roughly, querying U at some fixed y is like querying the squared Fourier coefficient $\gamma_y^{(S)}$.
What happens when we apply the diagonal matrix

$$\sum_S \gamma_y^{(S)} |\mathrm{Fock}(S)\rangle\langle\mathrm{Fock}(S)|\ ?$$

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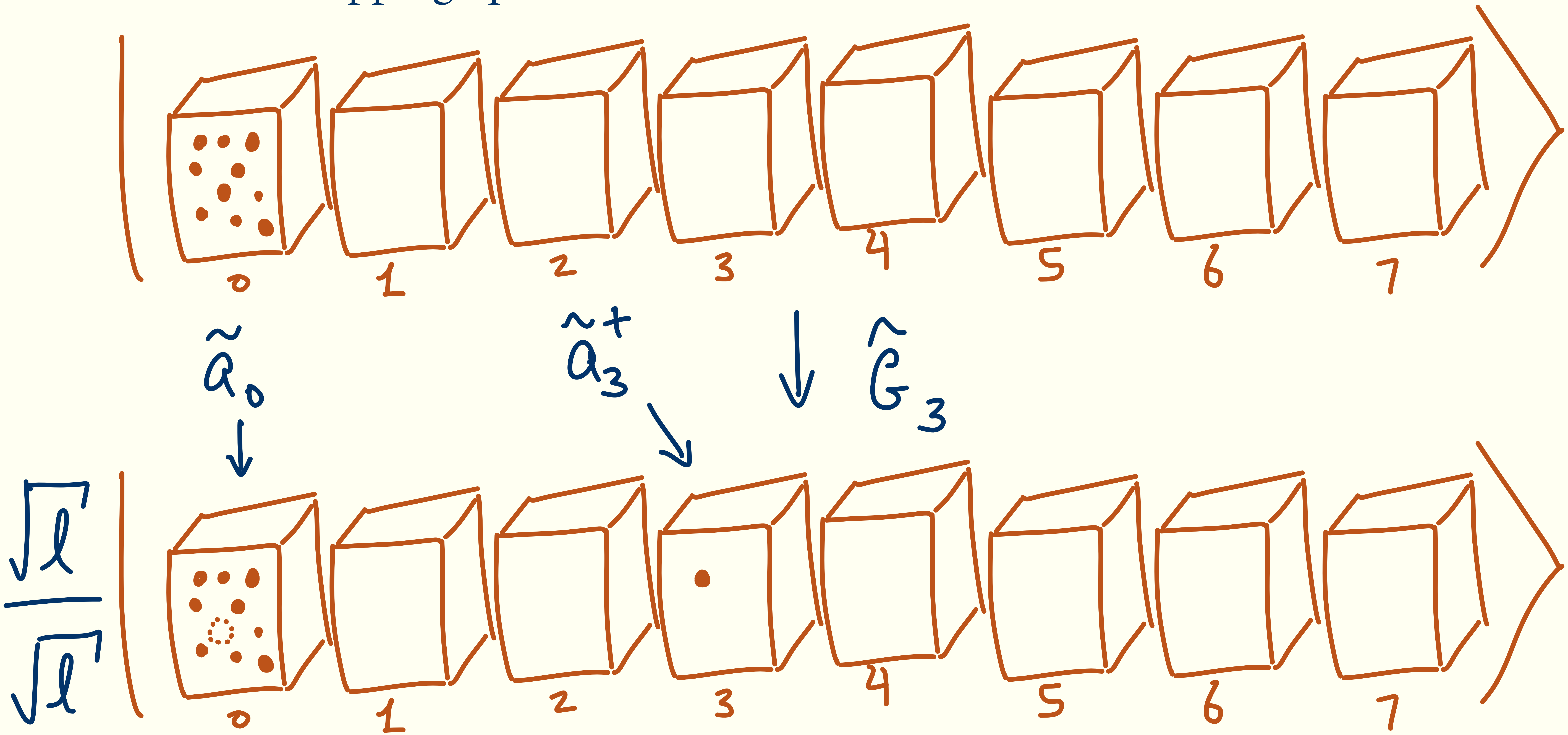
Let's define the momentum hopping and double hopping operators

$$\widetilde{G}_y = \frac{1}{\sqrt{\ell}} \sum_{x\in\{0,1\}^n} \widetilde{a}^{\dagger}_{x\oplus y}\widetilde{a}_x \quad \text{and} \quad \widetilde{H}_y = \frac{1}{\ell} \sum_{x,x'\in\{0,1\}^n} \widetilde{a}^{\dagger}_{x\oplus y}\widetilde{a}^{\dagger}_{x'\oplus y}\widetilde{a}_x\widetilde{a}_{x'}$$

Compressed oracles for Strong: Action of U

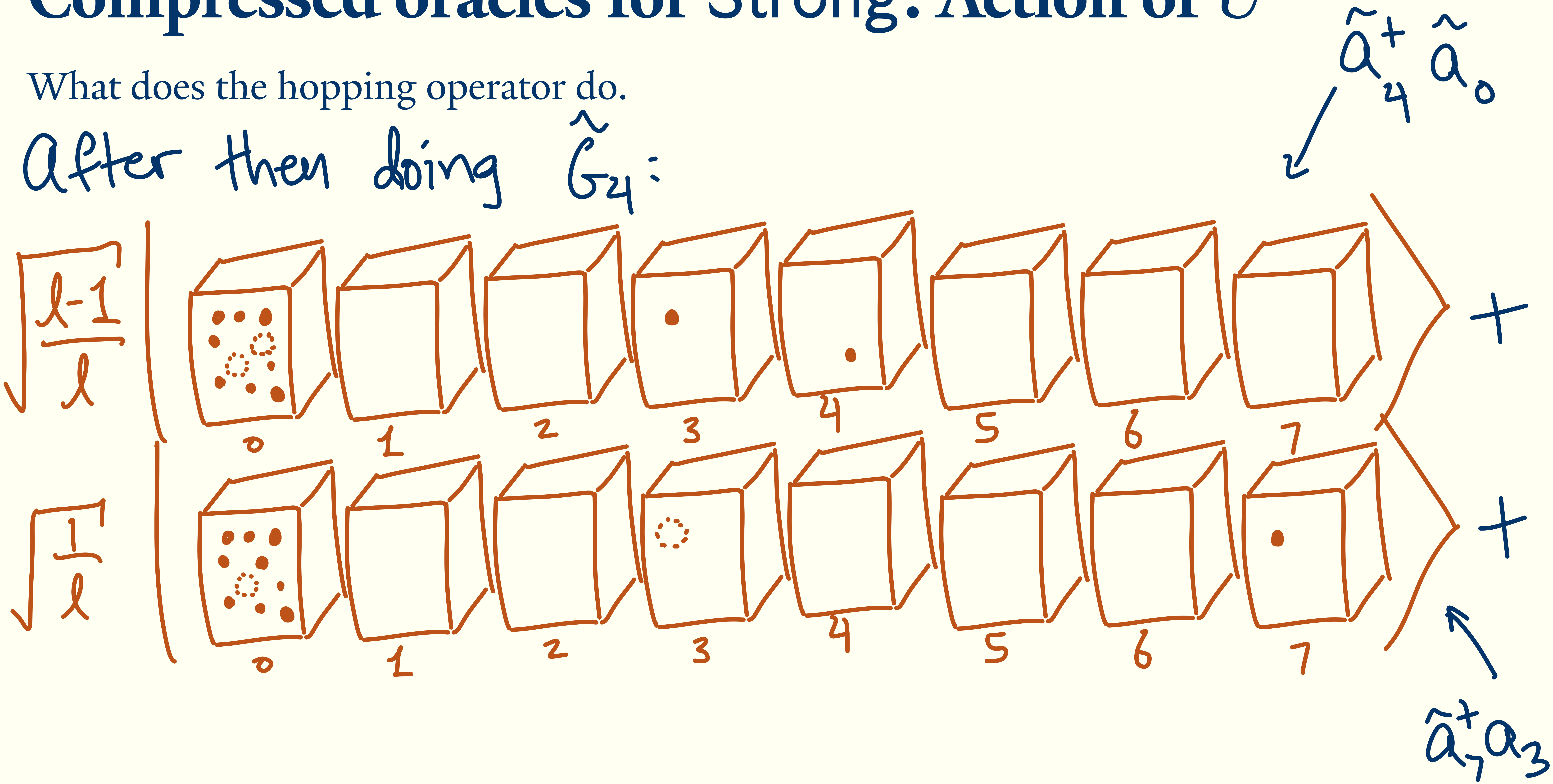
What does the hopping operator do.



Compressed oracles for Strong: Action of U

What does the hopping operator do.

After then doing $\tilde{G}_4$:



Compressed oracles for Strong: Action of U

Claim: The diagonal matrix that applies the squared Fourier coefficient is actually:

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Compressed oracles for Strong**: Action of** U

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Proof: We can expand out a position Fock state in the momentum basis and directly compute the action of the hopping operator (the double hopping is the square):

$$\widetilde{G}_y \hat{a}^\dagger_{s_1} \ldots \hat{a}^\dagger_{s_\ell} |\mathrm{vac}\rangle = \widetilde{G}_y \sum_{t_1,\ldots,t_\ell} \left(\prod_i (-1)^{t_i \cdot s_i} \right) \widetilde{a}^\dagger_{t_1} \ldots \widetilde{a}^\dagger_{t_\ell} |\mathrm{vac}\rangle$$

When we apply the hop and re-index the sum, we see that we just get a phase kickback!

Compressed oracles for Strong: Summary

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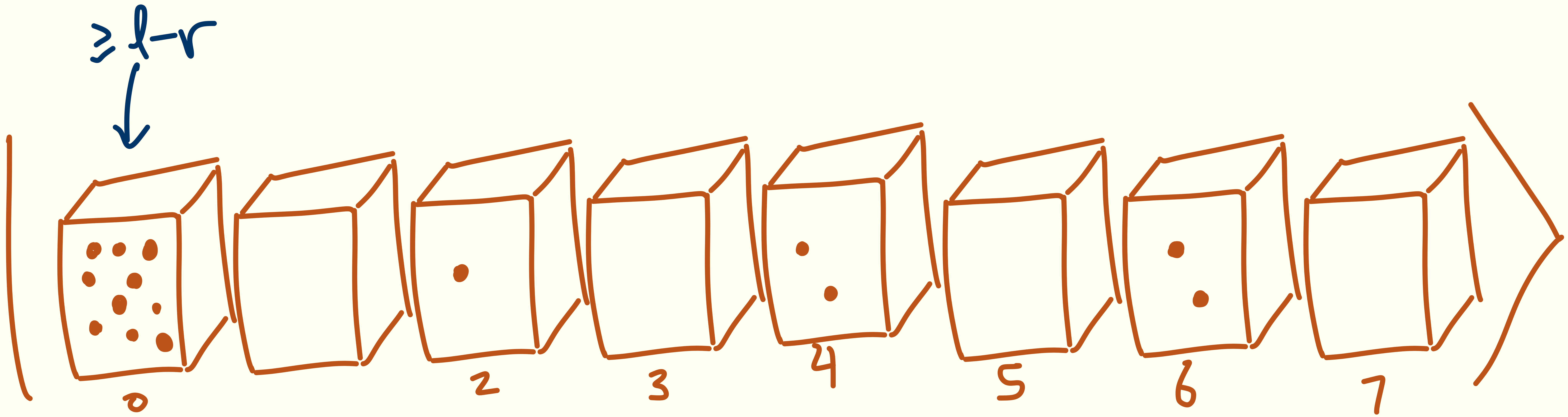
How does this let us prove a sampling probability upper bound?

Quasi-even condensates

A (r, o)-quasi-even condensate is a momentum Fock state $|\ell_0, \ldots, \ell_{2^n}\rangle$ that satisfies:

Condensate: $\ell_0 \geq \ell - r$, i.e., almost all of the bosons are in their initial position.



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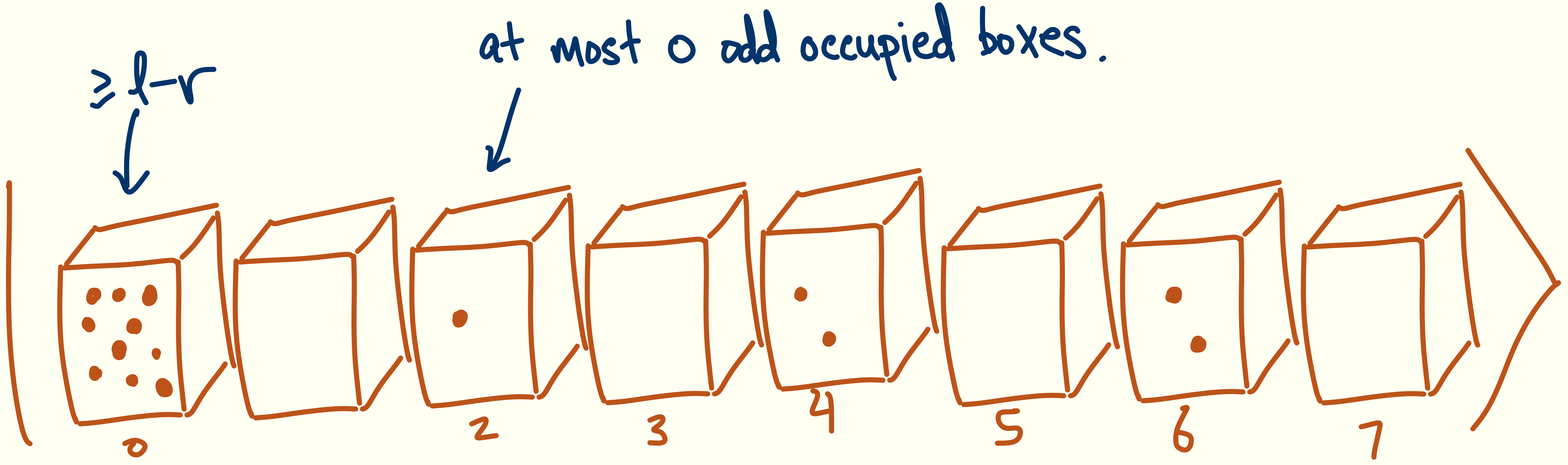
Quasi-even: At most o of the non-zero indices are odd.



Sampling bounds on quasi-even condensates

Claim: Let $|\psi\rangle$ be a state that is supported entirely on (r, o)-quasi-even condensate, then the following bound holds for all collections $z_1, \ldots, z_v \in \{0,1\}^n$:

$$\langle\psi|n_{z_1}\ldots, n_{z_\ell}|\psi\rangle \leq \left(\text{poly}(v, r) \cdot \frac{\sqrt{\ell}}{2^{n/4}}\right)^v$$

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This number upper bounds the sampling success probability (applying Markov's inequality). If we knew that the algorithm's purified state was supported only on quasi-even condensates, we would be done.

Sampling bounds on quasi-even condensates

Intuition for why quasi-evenness is the right notion:

- Imagine the position shift operator $\mathrm{Shift}_x^\dagger \cdot a_y^\dagger \cdot \mathrm{Shift}_x = a_{x \oplus y}^\dagger$.

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- This means that bosons in the condensate (0-momentum) and paired up are spread out uniformly among the positions (relative to the adversary's state), and therefore hard to guess.

Sampling bounds on quasi-even condensates

The final step is to show that an adversary querying the purified U is supported mostly on quasi-even condensates.

Roughly: The double hopping operator picks random bosons, so as long as it touches a r-condensate, most of its “weight” is two bosons from 0 to y momentum (on query y).

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After T-queries, we would expect to have a $\sim 2T$-condensate, and fewer than $v/4$ unpaired bosons, except with probability roughly $\left(vT^3\sqrt{\ell}/2^{n/4}\right)^v$.

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- Applying a flat polynomial approximation, we can show that these are close to polynomial in $\widetilde{G}_y^2$ whose degree is $\sim T^{10}$, which then gives us a condensate.
- But, this polynomial is not necessarily bounded anymore, so we need to bring in tools from perturbation theory (the Dyson series) to prove the quasi-even property.

Main theorems

Theorem 1: For all $v > 0$, and all quantum query algorithms making $T = T(n)$ queries to a set membership oracle for U, the probability, over Strong, that the algorithm outputs v distinct points from S is at most

$$\leq \left(\frac{\mathrm{poly}(v, T)}{\mathrm{poly}(2^n)}\right)^v.$$

Theorem 2: If there exists a QCMA algorithm, making $t = t(n)$ queries to (S, U) and taking a witness of length $q = q(n)$, then for all $0 < v < \ell/100$, there is a query algorithm making vt queries to U that outputs v distinct points from S with probability

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When $v \sim 1000q$, we get a contradiction

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  → This removal of structure allowed us to understand queries to the Fourier transform of an oracle way better than we could before!
- **Much more work is needed!**
  → Understanding oracles with structure seems to require an understanding that structure, seem to be annoying to deal with using general methods.
  → To understand other oracles (expander mixing problem, Yamakawa-Zhandry, etc.), we will need more specific tools, or a big leap in understanding of quantum algorithms.

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- **Is there a connection to the Aaronson-Ambainis conjecture?**
  → Both Liu-Mutreja-Yuen'24 and Zhandry'24 showed that there is a connection between QCMA versus QMA and pseudorandomness against quantum algorithms.
  → Our proof didn't say anything about this, but could you use our techniques?

Thanks for listening!